Constrained and stochastic variational principles for dissipative equations with advected quantities

Xin Chen[1], Ana Bela Cruzeiro[2], Tudor S. Ratiu[3]

Abstract

We develop symmetry reduction for material stochastic Lagrangian systems with advected quantities whose configuration space is a Lie group, obtaining deterministic constrained variational principles and dissipative equations of motion in spatial representation. We discuss in detail the situation when the group in the general theory is a group of diffeomorphisms and derive, as an application, an MHD system for viscous compressible fluids.

1 Introduction

The goal of this paper is to develop a Lagrangian symmetry reduction process for a large class of stochastic systems with advected parameters. The general theory, which yields deterministic constrained variational principles and deterministic reduced equations of motion, is developed for finite dimensional systems. The resulting abstract equations then serve as a template for the study of infinite dimensional stochastic systems, for which the rigorous analysis has to be carried out separately. The example of dissipative compressible magnetohydrodynamics equations is treated in detail.

The dynamics of many conservative physical systems can be described geometrically taking advantage of the intrinsic symmetries in their material description. These symmetries induce Noether conserved quantities and allow for the elimination of unknowns, producing an equivalent system consisting of new equations of motion in spaces with

[1]Department of Mathematics, Shanghai Jiao Tong University, 200240 Shanghai, P.R. China. chenxin217@sjtu.edu.cn

[2]Grupo de Física-Matemática and Dep. de Matemática, Instituto Superior Técnico, Univ. Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal. Partially supported by project PTDC/MAT/120354/2010 from the Portuguese F.C.T. abcruz@math.tecnico.ulisboa.pt

[3]Section de Mathématiques, École Polytechnique Fédérale de Lausanne, CH–1015 Lausanne, Switzerland. Partially supported by NCCR SwissMAP and grant 200020-126630, both from the Swiss National Science Foundation. tudor.ratiu@epfl.ch

less variables and a non-autonomous ordinary differential equation, called the reconstruction equation. This geometric procedure is known as reduction, a method that is ubiquitous in symplectic, Poisson, and Dirac geometry and has wide applications in theoretical physics, quantum and continuum mechanics, control theory, and various branches of engineering. For example, in continuum mechanics, the passage from the material (Lagrangian) to the spatial (Eulerian) or convective (body) description is a reduction procedure. Of course, depending on the problem, one of the three representations may be preferable. However, it is often the case, that insight from the other two representations, although apparently more intricate, leads to a deeper understanding of the physical phenomenon under consideration and is useful in the description of the dynamics.

A simple example in which the three descriptions are useful and serve different purposes is free rigid body dynamics (e.g., [62, Section 15]). If one is interested in the motion of the attitude matrix, the material picture is appropriate. The classical free rigid body dynamics result, obtained by applying Hamilton's standard variational principle on the tangent bundle of the proper rotation group $SO(3)$, states that the attitude matrix describes a geodesic of a left invariant Riemannian metric on $SO(3)$, characterized by the mass distribution of the body. However, as shown already by Euler, the equations of motion simplify considerably in the convective (or body) picture because the total energy of the rotating body, which in this case is just kinetic energy, is invariant under left translations. The convective description takes place on the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ and is given by the classical Euler equations for a free rigid body. Finally, the spatial description comes into play, because the spatial angular momentum is conserved during the motion and is hence used in the description of the rigid body motion.

The present paper uses exclusively Lagrangian mechanics, where variational principles play a fundamental role since they produce the equations of motion. In continuum mechanics, the variational principle used in the material description is the standard Hamilton principle producing curves in the configuration space of the problem that are critical points of the action functional. However, in the spatial and convective representations, if the configuration space of the problem is a Lie group, the induced variational principle requires the use of constrained variations, a fundamental result of Poincaré [69]; the resulting equations of motion are called today the *Euler-Poincaré equations* ([63], [62, Section 13.5], [14]). These equations have been vastly extended to include the motion of advected quantities ([13, 45, 46]) as well as affine ([36]) and non-commutative versions thereof that naturally appear in models of complex materials with internal structure ([37, 39]) and whose geometric description has led to the solution of a long-standing controversy in the nematodynamics of liquid crystals ([41, 42]). The Euler-Poincaré equations have also very important generalizations to problems whose configuration space is an arbitrary manifold and the Lagrangian is invariant under a Lie group action ([15]) as well as its extension to higher order Lagrangians

([32, 28, 29]). The Lagrange-Poincaré equations turn out to model the motion of spin systems ([31]), long molecules ([21, 30]), free boundary fluids and elastic bodies ([34]), as well as charged and Yang-Mills fluids ([39]). There is also a Lagrange-Poincaré theory for field theory ([38, 22]) and non-holonomic systems ([16]). The Lagrange-Poincaré equations also have interesting applications to Riemannian cubics and splines ([67]), the representation of images ([9]), certain classes of textures in condensed matter ([35]), and some control ([40]) and optimization ([33]) problems.

Variational principles play an important role in the design of structure preserving numerical algorithms. One discretizes both spatially and temporally such that the symmetry structure of the problem is preserved. Integrators based on a discrete version of Hamilton's principle are called variational integrators ([64]). The resulting equations of motion are the discrete Euler-Lagrange equations and the associated algorithm for classical conservative systems is both symplectic as well as momentum-preserving and manifests very good long time energy behavior; see [56, 57] for additional information. There are versions of such variational integrators for certain forced ([48]), controlled ([68]), constrained holonomic ([58, 60]), non-holonomic ([49]), non-smooth ([26, 19]), multiscale ([59, 73]), and stochastic ([6]) systems. In the presence of symmetry, these systems can be reduced. However, today a general theory of discrete reduction in all of these cases is still missing and is currently being developed. If the configuration space is a Lie group, the first discretization of symmetric Lagrangian systems appears in [66], motivated by problems in complete integrability; for an in-depth analysis of such problems see [72].

All the above mentioned systems, both in the smooth and discrete versions, should have various stochastic analogues, depending on what phenomenon is modeled. The basic idea is to start with variational principles, motivated by Feynman's path integral approach to quantum mechanics and also by stochastic optimal control. The latter has its origins in the foundational work of Bismut ([4, 5]) in the late seventies and in recent developments by Lázaro-Camí and Ortega ([52, 53, 54, 55]). Non-holonomic systems have been studied in the same spirit in [43]. All of this work investigated mainly stochastic perturbations of Hamiltonian systems. A very recent approach on the Lagrangian side, in Euler-Poincaré form, has been developed in [44], where both the position and the momentum of the system are (independently) randomly perturbed, as well as the Lagrangian.

The present paper takes a different approach and is situated in the perspective of Feynman's approach to quantum mechanics. Our Lagrangian is essentially (in the present paper, exactly) the classical one, but computed along stochastic paths, and their mean derivatives or drifts, in the spirit of [74], [75], [76]. We derive deterministic equations for the drifts of these Lagrangian processes which satisfy now (dissipative and deterministic) partial differential equations whose second order terms correspond to the generator of the martingale part of the underlying paths.

To illustrate our approach, let us consider a very simple example in the Euclidean

space $\mathbb{R}^n$. If $g(t)$ denote deterministic integral curves with velocity field u, we have $dg(t,x) = u(t, g(t,x))dt$, $g(0,x) = x$. When the paths g are perturbed by a Brownian motion $\sigma W_\omega(t)$ (σ a constant), then they are no longer differentiable but we can still write $dg_\omega(t,x) = \sigma dW(t) + u(t, g_\omega(t,x))dt$, $g_\omega(0,x) = x$, valid almost everywhere in ω, as long as we interpret the differential d in sense of the Itô calculus. Heuristically, the increments $g_\omega(t+\epsilon, x) - g_\omega(t,x)$ have normal distribution with mean $\epsilon u(t, g_\omega(t,x))$ and variance $\epsilon\sigma^2$. Dynamically, the paths g follow "in the mean" the directions determined by the vector field u (the drift) but are subjected to a dispersion due to the Brownian motion effect. More precisely, we have, almost-everywhere in ω,

$$u(t, g_\omega(t,x)) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(\mathbb{E}_t g_\omega(t+\epsilon, x) - g_\omega(t,x)\big)$$

where $\mathbb{E}_t$ denotes conditional expectation given the past at time t. So, u still corresponds to a (generalized) derivative. The generator of the process $g_\omega(t)$ is the operator defined by $Lf = \frac{\sigma^2}{2}\Delta f + (u \cdot \nabla)f$, i.e., we have

$$\mathbb{E}[f(g_\omega(t,x))] = f(x) + \mathbb{E}\int_0^t [Lf(g_\omega(s,x))]ds,$$

for every (compactly supported) smooth function f, where $\mathbb{E}$ denotes the expectation operator on the probability space whose points are ω. The Laplacian term accounts for the dispersion. In this sense, L deforms the time derivative along the classical flow with velocity field u.

Using such stochastic processes as "Lagrangian paths" one can derive variational principles extending those of classical mechanics. The critical points of action functionals, involving the above mentioned Lagrangians, solve almost surely the equations of motion. Those reduce to classical (Euler-Lagrange) equations in the limit $\sigma = 0$, by construction. It follows, in particular, that as in this classical limit case, the Lagrangian encodes most of the qualitative properties of the (stochastic) system.

In infinite dimensions, the Navier-Stokes equation was obtained in this manner, in the usual Eulerian (spatial) description ([2, 3, 17]), via a stochastic Euler-Poincaré reduction on the group of volume preserving diffeomorphisms. The Lagrangian variables correspond to semimartingales. In this paper, we derive Euler-Poincaré equations for stochastic processes defined on semidirect product Lie algebras and give the associated deterministic constrained variational principle. In other words, we develop the semidirect Euler-Poincaré reduction for a large class of stochastic systems.

Plan of the paper. In Section 2, we recall some basic probability notions necessary for the rest of the paper and give the definition of the generalized derivative for topological group valued semimartingales. Section 3 contains the first main result of the paper, namely the stochastic semidirect product Euler-Poincaré reduction for finite dimensional Lie groups, both in left and right-invariant versions. We give the deterministic variational principle and the reduced equations of motion. Section 4 presents

the second main result of the paper, the reduction from the material to the spatial representation of the stochastic compressible magnetohydrodynamics equations. The stochastic reduction process recovers the standard deterministic equations in Eulerian representation.

2 The derivative for semimartingales

In this section, as in [3], we introduce the notion of generalized derivative for semimartingales taking values on some topological groups. As we shall see later, the same definition holds if the semimartingale has values in some infinite dimensional group, such as the diffeomorphism group.

2.1 Some probability notions

We review in this subsection some basic notions of stochastic analysis on Euclidean spaces. We recall the concepts omitting the proofs, which can be found, for example in [47].

We denote $\mathbb{R}^+ := [0, \infty[$. Let $(\Omega, \mathcal{P}, P)$ be a probability space. Suppose we are given a family $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ of sub-σ-algebras of $\mathcal{P}$ which is non-decreasing (namely, $\mathcal{P}_s \subset \mathcal{P}_t$ for $0 \leq s \leq t$) and right-continuous, i.e., $\cap_{\epsilon>0}\mathcal{P}_{t+\epsilon} = \mathcal{P}_t$ for all $t \in \mathbb{R}^+$. We then say that the probability space is endowed with a *non-decreasing filtration* $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$.

A stochastic process $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $(\mathcal{P}_t)$-*adapted* if $X(t)$ is $\mathcal{P}_t$-measurable for every t. Typically, filtrations describe the past history of a process: one starts with a process X and defines $\mathcal{P}_t := \cap_{\epsilon>0}\sigma\{X(s), 0 \leq s \leq t\}$. Then the process X will be automatically $(\mathcal{P}_t)$-adapted.

A stochastic process $M : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is a *martingale* if

(i) $\mathbb{E}|M(t,\omega)| < \infty$ for all $t \geq 0$;
(ii) $M(t,\omega)$ is $(\mathcal{P}_t)$-adapted;
(iii) $\mathbb{E}_s(M(t,\omega))) = M(s,\omega)$ a.s. for all $0 \leq s < t$.

In the above definition, $\mathbb{E}$ denotes the expectation of the random variable, whereas $\mathbb{E}_s(M(t,\omega)) := \mathbb{E}(M(t,\omega)|\mathcal{P}_s)$, for each $s \geq 0$, is the conditional expectation of the random variable $M_\omega(t)$, $t > s$, relative to the $\sigma-$algebra $(\mathcal{P}_s)$, i.e., $\Omega \ni \omega \mapsto \mathbb{E}_s(M_\omega(t)) \in \mathbb{R}$ is a $\mathcal{P}_s$-measurable function satisfying

$$\int_A \mathbb{E}_s(M(t,\omega))dP(\omega) = \int_A M(t,\omega)dP(\omega), \quad \forall A \in \mathcal{P}_s.$$

Thus, condition (iii) is equivalent to $\mathbb{E}((M_\omega(t) - M_\omega(s))\chi_A) = 0$ for all $A \in \mathcal{P}_s$ and all $t, s \in \mathbb{R}$ satisfying $t > s \geq 0$, where χ_A is the characteristic function of the set A.

From now on we will often drop the probability variable ω in the notations. In this paper we shall only consider processes defined in a compact time intervals $[0,T]$ which are continuous (i.e., continuous with respect to the time variable t for almost all $\omega \in \Omega$).

If a martingale M is continuous and $\mathbb{E}(M_\omega(t)^2) < \infty$ for all $t \geq 0$, we say that M has a *quadratic variation* $\{[\![M,M]\!]_t, t \in [0,T]\}$ if $M^2(t) - [\![M,M]\!]_t$ is a martingale, and $[\![M,M]\!]_t$ is a continuous, $\mathcal{P}_t$-adapted, a.s. non-decreasing process with $[\![M,M]\!]_0 = 0$. Such a process is unique and coincides with the following limit (convergence in probability),

$$\lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} (M(t_{i+1}) - M(t_i))^2$$

where σ_n is a partition of the interval $[0,t]$ and the mesh converges to zero as $n \to \infty$.

Actually, the definition of the quadratic variation requires only right-continuity of M. Moreover, for two martingales M and N, under the same assumptions and conventions as given above, one can also define their *covariation*

$$[\![M,N]\!]_t := \lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} (M(t_{i+1}) - M(t_i))(N(t_{i+1}) - N(t_i)),$$

which extends the notion of quadratic variation. Clearly,

$$2[\![M,N]\!]_t = [\![M+N, M+N]\!]_t - [\![M,M]\!]_t - [\![N,N]\!]_t.$$

More generally, one can consider local martingales. A *stopping time* is a random variable $\tau : \Omega \to \mathbb{R}^+$ such that for all $t \geq 0$ the set $\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{P}_t$. Then a stochastic process M is a *local martingale* if there exists a sequence of stopping times $\{\tau_n, n \geq 1\}$, such that $\lim_{n\to\infty} \tau_n = \infty$ a.s., and $M^n(t) := M(t \wedge \tau_n)$ is a square integrable martingale for all $n \geq 1$, where $t \wedge \tau_n := \min(t, \tau_n)$. We then define $[\![M,M]\!]_t := [\![M^n, M^n]\!]_t$ if $t \leq \tau_n$.

A real-valued *Brownian motion* is a continuous martingale $W(t)$, $t \in [0,T]$, such that $W^2(t) - t$ is a martingale; or, equivalently, such that $[\![W,W]\!]_t = t$.

A stochastic process $X : \Omega \times [0,T] \to \mathbb{R}$ is a *semimartingale* if, for every $t \geq 0$, it can be decomposed into a sum

$$X(t) = X(0) + M(t) + A(t),$$

where M is a local martingale with $M(0) = 0$ and A is a càdlàg adapted process of locally bounded variation with $A(0) = 0$ a.s. (the definition requires that A be right-continuous with left limits at each $t \geq 0$; however, we consider only processes that are continuous in time, which is a standard assumption throughout this paper).

More generally, M can be only a local martingale and A a locally bounded variation process. The definition of a *local semimartingale* follows the one given above for local martingales.

For a (local) semimartingale we define $[\![X,X]\!]_t := [\![M,M]\!]_t$.

Martingales and, in particular, Brownian motion, are not (a.s.) differentiable in time (unless they are constant); therefore one cannot integrate with respect to martingales as one does with respect to functions of bounded variation. We recall the definition of the two most commonly used stochastic integrals, the Itô and the Stratonovich integrals.

If X and Y are continuous real-valued semimartingales such that

$$\mathbb{E}\left(\int_0^T |X(t)|^2 dt + \int_0^T |Y(t)|^2 dt\right) < \infty,$$

the *Itô stochastic integral* in the time interval $[0,t]$, $0 < t \leq T$, with respect to Y is defined as the limit in probability (if the limit exists) of the sums

$$\int_0^t X(s)dY(s) = \lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} X(t_i)(Y(t_{i+1}) - Y(t_i))$$

where σ_n is a partition of the interval $[0,t]$ with mesh converging to zero as $n \to \infty$. Moreover, when Y is a martingale such that $\mathbb{E}\left(\int_0^T |X(t)|^2 d[\![Y,Y]\!]_t\right) < \infty$, then $\int_0^t X(s)dY(s),\ t \in [0,T]$, is also a martingale.

The *Stratonovich stochastic integral* is defined by

$$\int_0^t X(s)\delta Y(s) = \lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} \frac{(X(t_i) + X(t_{i+1}))}{2}(Y(t_{i+1}) - Y(t_i))$$

whenever such limit exists.

These integrals do not coincide in general, even though X is a continuous process, due to the lack of differentiability of the paths of Y. The Itô and the Stratonovich integrals are related by

$$\int_0^t X_s \delta Y_s = \int_0^t X_s dY_s + \frac{1}{2}\int_0^t d[\![X,Y]\!]_s \tag{2.1}$$

If $f \in C^2(\mathbb{R})$, Itô's formula states that

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[\![X,X]\!]_s \tag{2.2}$$

This formula, for Stratonovich integrals, reads,

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\delta X(s)$$

One advantage of Stratonovich integrals is that they allow the use of the same rules as those of the standard deterministic differential calculus. On the other hand an Itô integral with respect to a martingale M is again a martingale, a very important property. For example, we have, as an immediate consequence, that $\mathbb{E}_s \int_s^t X(r)dM(r) = 0$ for all $0 \leq s < t$.

In higher dimensions, the difference between the Stratonovich and the Itô integral in Itô's formula is given in terms of the Hessian of f (see subsection 2.2). In fact, suppose that X be a $\mathbb{R}^d$-valued semimartingale; then Itô's formula in d-dimensions (see also (2.2)) states that, for every $f \in C^2(\mathbb{R}^d)$,

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{i=1}^d \int_0^t \partial_i f(X(s))dX^i(s) + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{i,j}(X(s))d[\![X^i, X^j]\!]_s \\ &= f(X(0)) + \sum_{i=1}^d \int_0^t \partial_i f(X(s))\delta X^i(s) \end{aligned} \tag{2.3}$$

For Brownian motions W^i, $i = 1\ldots, k$, we have

$$d[\![W^i, W^j]\!]_t = \delta_{ij}dt \tag{2.4}$$

where δ_{ij} denotes the Kronecker delta symbol, and, as the covariation of semimartingales is determined by their martingale parts, the following identities hold (see, e.g., [47]),

$$d[\![W^i, \iota]\!]_t = 0 \quad \forall i = 1, \ldots, d, \qquad d[\![\iota, \iota]\!]_t = 0, \tag{2.5}$$

where $\iota(t) = t$ is the identity (deterministic) function.

2.2 The generalized derivative for (topological) group valued semimartingales

Let G denote a topological group, endowed with a Banach manifold structure (possibly infinite dimensional) whose underlying topology is the given one, such that all left (or right) translations L_g (resp. R_g) by arbitrary $g \in G$ are smooth maps; $L_g h := gh$, $R_g h := hg$, for all $g, h \in G$. Given $v \in T_eG$, denote by v^L (resp. v^R) the left (resp. right) invariant vector field whose value at the neutral element e of G is v, i.e., $v^L(g) := T_eL_g v$ (resp. $v^R(g) := T_eR_g v$), where $T_eL_G : T_eG \to T_gG$ is the tangent map (derivative) of L_g (and similarly for R_g). The operation $[v_1, v_2] := \left[v_1^L, v_2^L\right](e)$, for any $v_1, v_2 \in T_eG$, defines a (left) Lie bracket on T_eG. When working with right invariant vector fields, we shall still use, formally, the *left* Lie bracket defined above, i.e., we shall never work with right Lie algebras; the bracket defined by right invariant vector

fields is equal to the negative of the left Lie bracket defined above. Denote, as usual, by $\operatorname{ad}_u v := [u, v]$ the adjoint action of T_eG on itself and by $\operatorname{ad}_u^* : T_e^*G \to T_e^*G$ its dual map (the coadjoint action of T_eG on its dual T_e^*G).

Suppose that ∇ is a left invariant linear connection on G, i.e., $\nabla_{v_1^L} v_2^L$ is a left invariant vector field, for any $v_1, v_2 \in T_eG$. Then we define $\nabla_{v_1} v_2 := \nabla_{v_1^L} v_2^L(e)$ for all $v_1, v_2 \in T_eG$. If right translation is smooth, in all the definitions above, we can replace left by right translation in a similar way. The left invariant connection ∇ is torsion free if and only if

$$\nabla_{v_1} v_2 - \nabla_{v_2} v_1 = [v_1, v_2], \quad \text{for all} \quad v_1, v_2 \in T_eG$$

as an easy verification shows. For a fixed $g_1 \in G$, let $T_{g_2}L_{g_1} : T_{g_2}G \to T_{g_1 g_2}G$ be the *tangent map* (or derivative) of L_{g_1} at the point $g_2 \in G$.

Let G be endowed with a left invariant linear torsion free connection ∇. The corresponding *Hessian* $\operatorname{Hess} f(g) : T_gG \times T_gG \to \mathbb{R}$ of $f \in C^2(G)$ at $g \in G$ is defined by

$$\operatorname{Hess} f(g)\Big(v_1, v_2\Big) := \tilde{v}_1 \tilde{v}_2 f(g) - \nabla_{\tilde{v}_1} \tilde{v}_2 f(g), \quad v_1, v_2 \in T_gG, \tag{2.6}$$

where $\tilde{v}_i,\ i = 1, 2$, are arbitrary smooth vector fields on G such that $\tilde{v}_i(g) = v_i$. Since the connection is torsion free, $\operatorname{Hess} f(g)$ is a symmetric $\mathbb{R}$-bilinear form on each T_gG. In addition, $\operatorname{Hess} f = \nabla^2 f = \nabla \mathbf{d} f$ (see, e.g., [1] or [23]) is the covariant derivative associated with ∇ of the one-form $\mathbf{d}f$, where $\mathbf{d}$ denotes the exterior differential.

A *semimartingale with values in* G is a stochastic process $g : \Omega \times \mathbb{R}^+ \to G$ such that, for every function $f \in C^2(G)$, $f \circ g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a real-valued semimartingale, as introduced in subsection 2.1 (see, e.g., [23] for the case of finite dimensional Lie groups).

A semimartingale with values in G is a ∇-*(local) martingale* if

$$t \longmapsto f(g_\omega(t)) - f(g_\omega(0)) - \frac{1}{2} \int_0^t \operatorname{Hess} f(g_\omega(s)) d[\![g_\omega, g_\omega]\!]_s ds$$

is a real-valued (local) martingale, where $[\![g_\omega, g_\omega]\!]_t$ is the quadratic variation of g_ω. If G is a finite dimensional Lie group, then we have the following expression

$$d[\![g_\omega, g_\omega]\!]_t := d \left[\!\!\left[\int_0^\cdot \mathbf{P}_s^{-1} \delta g_\omega(s), \int_0^\cdot \mathbf{P}_s^{-1} \delta g_\omega(s) \right]\!\!\right]_t,$$

where $\mathbf{P}_t : T_{g_\omega(0)}G \to T_{g_\omega(t)}G$ is the (stochastic) parallel translation along the (stochastic) curve $t \mapsto g_\omega(t)$ associated with the connection ∇, see e.g., [23] or [47]. Moreover, for some infinite dimensional groups G (for example the diffeomorphism group on a

torus), the quadratic variation is also well defined: we refer the readers to [3, 17] for details (see also Section 4 of this paper).

For a G-valued semimartingale $g_\omega(\cdot)$, suppose there is a, possibly random, process $\mathbf{v}(t,\omega) : (t,\omega) \mapsto TG$, such that $\mathbf{v}(t) \in T_{g_\omega(t)}G$ a.s., and for every $f \in C^\infty(G)$, the process

$$N_t^f := f(g_\omega(t)) - f(g_\omega(0))) - \frac{1}{2}\int_0^t \mathrm{Hess} f(g_\omega(s)))d[\![g_\omega, g_\omega]\!]_s - \int_0^t \langle \mathbf{d}f(g_\omega(s)), \mathbf{v}_\omega(s)\rangle\, ds$$

is a real-valued local martingale. Then we define the ∇-*generalized derivative* of $g_\omega(t)$ by

$$\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} := \mathbf{v}_\omega(t). \tag{2.7}$$

We remark that by (2.7), the ∇-generalized derivative is well defined for semimartingales with values in a finite dimensional Lie group as well as in some infinite dimensional groups (the diffeomorphism group on a torus for example), see e.g. [3]. Moreover, when G is a finite dimensional compact Lie group, we have the following equality (see [23])

$$\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} := \mathbf{P}_t\left(\lim_{\epsilon\to 0}\mathbb{E}_t\left[\frac{v_\omega(t+\epsilon) - v_\omega(t)}{\epsilon}\right]\right) \in T_{g_\omega(t)}G, \tag{2.8}$$

where

$$v_\omega(t) = \int_0^t \mathbf{P}_s^{-1}\delta g_\omega(s) \in T_eG.$$

The conditional expectation $\mathbb{E}_t$ plays a major rôle in the definition of the ∇-generalized derivative, since it eliminates the martingale part of the semimartingale. Therefore, the velocities are given by the drift (the bounded variation part) and the diffusion part (the martingale) can be seen a stochastic perturbation; in other words, the drift determines the directions where the particles flow, the martingale part describes their random fluctuations.

Moreover, according to the definition, if a G-valued semimartingale $g_\omega(t)$ satisfies $\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} = 0$, then $g_\omega(t)$ is a ∇-martingale.

In Euclidean space, endowed with the standard Levi-Civita connection of the constant Riemannian metric given by the inner product, (2.7) coincides with the definition of the generalized derivative for the Euclidean valued semimartingale considered in [17] and [74], [76].

Given $\mathbb{R}^k$-valued Brownian motion $W_\omega(t) = (W_\omega^1(t), ..., W_\omega^k(t)),\ t \in [0,T]$, vectors $H_i \in T_eG$, $1 \le i \le k$, and curve $u \in C^1([0,T]; T_eG)$, we consider the following Stratonovich SDE on G,

$$\begin{cases} dg_\omega(t) = T_eL_{g_\omega(t)}\left(\sum_{i=1}^k H_i\delta W_\omega^i(t) - \frac{1}{2}\nabla_{H_i}H_i dt + u(t)dt\right), \\ g_\omega(0) = e; \end{cases} \tag{2.9}$$

here, for each $v, w \in T_eG$, we recall that $\nabla_v w \in T_eG$ by $\nabla_v w := \left(\nabla_{\widetilde{v}}\widetilde{w}\right)(e)$, where $\widetilde{v}$ and $\widetilde{w}$ are the left invariant vector fields (or right invariant, if the connection is right invariant) on G, such that $\widetilde{v}(e) = v$, $\widetilde{w}(e) = w$.

Since $d[\![(T_eL_{g_\omega(t)}H_i), W^i_\omega(t)]\!]_t = T_eL_{g_\omega(t)}(\nabla_{H_i}H_i)dt$, formula (2.1) shows that equation (2.9) is equivalent to

$$
\left\{\begin{array}{l} dg_\omega(t) = T_eL_{g_\omega(t)}\left(\sum_{i=1}^k H_i dW^i_\omega(t) + u(t)dt\right), \\ g_\omega(0) = e. \end{array}\right. \tag{2.10}
$$

If G is a finite dimensional Lie group, there exists a unique strong solution for (2.9) (c.f. [47], [23]). When G is the diffeomorphism group on a torus a weak solution still exists ([3], [17]).

Applying Itô's formula for G-valued semimartingales (see [23] for the case where G is finite dimensional and [3], section 4.2 for the case where G is the diffeomorphism group on a torus), for every $f \in C^2(G)$ we have,

$$
f(g_\omega(t)) = f(g_\omega(0)) + N^f_t + \frac{1}{2}\int_0^t \mathrm{Hess}f(g_\omega(s))d[\![g_\omega, g_\omega]\!]_s + \int_0^t T_eL_{g_\omega(s)}(u(s))f(g_\omega(s))ds
$$

where N^f_t is a martingale. Actually, this last equality, valid for each $f \in C^2(G)$, is a characterization of the solution of the stochastic differential equation (2.9) (or (2.10)), in a weak sense.

Clearly, when G is a finite dimensional Lie group, by definition (2.7), we have

$$
\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} = T_eL_{g_\omega(t)}u(t). \tag{2.11}
$$

3 Stochastic semidirect product Euler-Poincaré reduction

In this section, inspired by [3], we generalize the deterministic semidirect product Euler-Poincaré reduction, formulated and developed in in [45], to the stochastic setting.

Left invariant version. Let U be a vector space, U^* its dual, and $\langle\cdot,\cdot\rangle_U : U^*\times U \to \mathbb{R}$ the duality pairing. Let G be a group endowed with a manifold structure making it into a topological group whose left translation is smooth. As discussed in subsection 2.2, the tangent space T_eG to G at the identity element $e \in G$ is a Lie algebra. Assume that G has a left representation on U; therefore, there are naturally induced left representations of the group G and the Lie algebra T_eG on U and U^*. All actions will be denoted by concatenation. Let $\langle\cdot,\cdot\rangle_{T_eG} : T^*_eG \times T_eG \to \mathbb{R}$ be the duality pairing between T^*_eG and T_eG. Define the operator $\diamond : U \times U^* :\to T^*_eG$ by

$$
\langle a \diamond \alpha, v\rangle_{T_eG} := -\langle v\alpha, a\rangle_U = \langle \alpha, va\rangle_U, \qquad v \in T_eG, \quad a \in U, \quad \alpha \in U^*. \tag{3.1}
$$

In fact, $a \diamond \alpha$ is the value at (a, α) of the momentum map $U \times U^* \to T_e^*G$ of the cotangent lifted action induced by the left representation of G on U.

Let $\mathscr{S}(G)$ denote the collection of G-valued continuous semimartingales defined on the time interval $t \in [0, T]$. Given a (left invariant) linear connection ∇ on G, a point $\alpha_0 \in U^*$, and a (Lagrangian) function $l : T_eG \times U^* \to \mathbb{R}$, define the *action functional* $J^{\nabla,\alpha_0,l} : \mathscr{S}(G) \times \mathscr{S}(G) \to \mathbb{R}_+$ by

$$J^{\nabla,\alpha_0,l}(g^1_\omega(\cdot), g^2_\omega(\cdot)) := \mathbb{E} \int_0^T l\left(T_{g^1_\omega(t)} L_{g^1_\omega(t)^{-1}} \frac{\mathscr{D}^\nabla g^1_\omega(t)}{dt}, \alpha(t)\right) dt, \tag{3.2}$$

where $g^1_\omega(\cdot), g^2_\omega(\cdot) \in \mathscr{S}(G)$ and

$$\alpha(t) := \mathbb{E}\big[\tilde{\alpha}_\omega(t)\big] \in U^*, \qquad \tilde{\alpha}_\omega(t) := g^2_\omega(t)^{-1}\alpha_0. \tag{3.3}$$

For every (deterministic) curve $\mathfrak{g}(\cdot) \in C^1([0,1]; T_eG)$ satisfying $\mathfrak{g}(0) = \mathfrak{g}(T) = 0$ and $\varepsilon \in [0, 1)$, let $e_{\varepsilon,\mathfrak{g}}(\cdot) \in C^1([0, T]; G)$ be the unique solution of the (deterministic) time-dependent differential equation on G

$$\begin{cases} \frac{d}{dt} e_{\varepsilon,\mathfrak{g}}(t) = \varepsilon T_e L_{e_{\varepsilon,\mathfrak{g}}(t)} \dot{\mathfrak{g}}(t), \\ e_{\varepsilon,\mathfrak{g}}(0) = e. \end{cases} \tag{3.4}$$

Note that this system implies $e_{0,\mathfrak{g}}(t) = e$ for all $t \in [0, T]$.

We say that $\big(g^1_\omega(\cdot), g^2_\omega(\cdot)\big) \in \mathscr{S}(G) \times \mathscr{S}(G)$ is a *critical point* of $J^{\nabla,\alpha_0,l}$ if for every (deterministic) curve $\mathfrak{g}(\cdot) \in C^1([0, 1]; T_eG)$ with $\mathfrak{g}(0) = \mathfrak{g}(T) = 0$, we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J^{\nabla,\alpha_0,l}\left(g^1_{\omega,\varepsilon,\mathfrak{g}}(\cdot), g^2_{\omega,\varepsilon,\mathfrak{g}}(\cdot)\right) = 0 \tag{3.5}$$

where

$$g^i_{\omega,\varepsilon,\mathfrak{g}}(t) := g^i_\omega(t) e_{\varepsilon,\mathfrak{g}}(t),\ t \in [0, T],\ i = 1, 2,\ \varepsilon \in [0, 1). \tag{3.6}$$

We emphasize the particular form of these deformations in the Lie group: they correspond to developments along deterministic directions $\mathfrak{g}(t)$.

Remark 3.1. As will be seen in the applications in the next section, the reason we choose two different semimartingales $(g^1_\omega(\cdot), g^2_\omega(\cdot))$ in the variational principle (3.5) is that the viscosity constants in the equations for the variables in T_eG and U^* may be different. $\diamondsuit$

From now on in this section we assume that both G and U are finite dimensional. Theorem 3.2 still holds for some infinite dimensional models, see section 4 for the case of the diffeomorphism group on the torus. We fix $H^m_j \in T_eG$, with $m = 1, 2$,

$j = 1, ..., k_m$, as well as $W_\omega^m(t) = (W_\omega^{m,1}(t), ..., W_\omega^{m,k_m}(t))$, $m = 1, 2$ independent $\mathbb{R}^{k_m}$ valued Brownian motions (notice that, in the infinite dimensional case, we may not be able to fix a priori the Brownian motions in order to solve the corresponding stochastic differential equations).

We consider semimartingales $g_\omega^m(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, defined by

$$\left\{ \begin{array}{l} dg_\omega^m(t) = T_e L_{g_\omega^m(t)} \left(\sum_{j=1}^{k_m} \left(H_j^m \delta W_\omega^{m,j}(t) - \frac{1}{2} \nabla_{H_j^m} H_j^m dt \right) + u(t)dt \right), \\ g_\omega^m(0) = e, \end{array} \right. \tag{3.7}$$

for some $u(\cdot) \in C^1([0,T]; T_eG)$. Note that $u(\cdot)$ is not given (and is the same for $m = 1, 2$); we shall see below that it is the solution of a certain equation when $(g_\omega^1(\cdot), g_\omega^2(\cdot))$ is critical for $J^{\nabla,\alpha_0,l}$.

In the theorem below we use the functional derivative notation. Let V be (a possibly infinite dimensional) vector space and V^* a space in weak duality $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ with V; in finite dimensions, V^* is the usual dual vector space, but in infinite dimensions it rarely is the topological dual. If $f : V \to \mathbb{R}$ is a smooth function, then the *functional derivative* $\frac{\delta f}{\delta a} \in V^*$, if it exists, is defined by $\lim_{\epsilon \to 0} \frac{f(a+\epsilon b) - f(a)}{\epsilon} = \left\langle \frac{\delta f}{\delta a}, b \right\rangle$ for all $a, b \in V$. Thus, in the theorem below $\frac{\delta l}{\delta u} \in T_e^*G$ and $\frac{\delta l}{\delta \alpha} \in U$ are the two partial functional derivatives of $l : T_eG \times U^* \to \mathbb{R}$.

Theorem 3.2. *Assume G is a finite dimensional Lie group endowed with a left invariant linear connection ∇. Let U be a finite dimensional left G-representation space and $l : T_eG \times U^* \to \mathbb{R}$ a smooth function. Suppose that the semimartingales $g_\omega^m(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, have the form* (3.7) *and their deformations are of the form* (3.6)*.*

(i) *Then $(g_\omega^1(\cdot), g_\omega^2(\cdot))$ is a critical point of $J^{\nabla,\alpha_0,l}$ (see* (3.2)*) if and only if the (non-random) curve $u(\cdot) \in C^1([0,T]; T_eG)$ coupled with $\alpha(\cdot) \in C^1([0,T]; U^*)$ (defined by* (3.3)*) satisfies the following* semidirect product Euler-Poincaré equation for stochastic particle paths:

$$\left\{ \begin{array}{l} \frac{d}{dt} \frac{\delta l}{\delta u} = \mathrm{ad}_{\tilde{u}^1}^* \frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha + K \left(\frac{\delta l}{\delta u} \right), \\ \frac{d}{dt} \alpha(t) = \frac{1}{2} \sum_{j=1}^{k_2} H_j^2 \left(H_j^2 \alpha(t) \right) - \tilde{u}^2(t) \alpha(t), \end{array} \right. \tag{3.8}$$

where $\frac{\delta l}{\delta u} \in T_e^*G$ and $\frac{\delta l}{\delta \alpha} \in U$ denote the two partial functional derivatives of l, the operation $\diamond$ is defined by (3.1)*, $\alpha(\cdot)$ by* (3.3)*,*

$$\begin{aligned} \tilde{u}^1(t) &:= u(t) - \frac{1}{2} \sum_{j=1}^{k_1} \nabla_{H_j^1} H_j^1, \\ \tilde{u}^2(t) &:= u(t) - \frac{1}{2} \sum_{j=1}^{k_2} \nabla_{H_j^2} H_j^2. \end{aligned} \tag{3.9}$$

and the operator $K : T_e^*G \to T_e^*G$ is defined for all $\mu \in T_e^*G$ and $v \in T_eG$ by

$$\langle K(\mu), v\rangle = -\left\langle \mu, \frac{1}{2}\sum_{j=1}^{k_1}\left(\nabla_{\mathrm{ad}_v H_j^1} H_j^1 + \nabla_{H_j^1}(\mathrm{ad}_v H_j^1)\right)\right\rangle. \tag{3.10}$$

(ii) *The first equation in* (3.8) *is equivalent to the* dissipative Euler-Poincaré variational principle

$$\delta \int_0^T l(u(t), \alpha(t))dt = 0 \tag{3.11}$$

on $T_eG \times U^*$ defined by $l : T_eG \times U^* \to \mathbb{R}$, for variations of the form

$$\begin{cases} \delta u = \dot{v} + [\tilde{u}^1, v] + K^*(v) = \dot{v} + [u, v] - \dfrac{1}{2}\left[\displaystyle\sum_{j=1}^{k_1} \nabla_{H_j^1} H_j^1, v\right] + K^*(v), \\ \delta\alpha = -v\alpha \end{cases} \tag{3.12}$$

where $v(t)$ is an arbitrary curve in T_eG vanishing at the endpoints, i.e., $v(0) = 0$, $v(T) = 0$. Note that this variational principle is constrained *and* deterministic*.*

Proof. (i) **Step 1.** We start by proving prove that $\alpha(t)$ satisfies the second equation in (3.8). We have

$$d\left(g_\omega^2(t)\right)^{-1} = -T_eR_{(g_\omega^2(t))^{-1}} T_{g_\omega^2(t)} L_{(g_\omega^2(t))^{-1}} dg_\omega^2(t)$$

and, replacing $dg_\omega^2(t)$ by its expression,

$$\begin{cases} d(g_\omega^2(t))^{-1} = T_eR_{(g_\omega^2(t))^{-1}} \left(\sum_{j=1}^{k_2}\left(-H_j^2 \delta W_\omega^{2,j}(t) + \frac{1}{2}\nabla_{H_j^2} H_j^2 dt\right) - u(t)dt\right), \\ g_\omega^2(0) = e, \end{cases} \tag{3.13}$$

We now derive the stochastic differential equation satisfied by $\tilde{\alpha}_\omega(t) := g_\omega^2(t)^{-1}\alpha_0$:

$$\begin{aligned} d\tilde{\alpha}_\omega(t) &= d\left(g_\omega^2(t)^{-1}\alpha_0\right) = \left[-T_{g_\omega^2(t)} L_{g_\omega^2(t)^{-1}} dg_\omega^2(t)\right] g_\omega^2(t)^{-1}\alpha_0 \\ &= -\sum_{j=1}^{k_2}\left(H_j^2\left(g_\omega^2(t)^{-1}\alpha_0\right)\delta W_\omega^{2,j}(t) - \left(\frac{1}{2}\nabla_{H_j^2} H_j^2\right)\left(g_\omega^2(t)^{-1}\alpha_0\right)dt\right) \\ &\quad - u(t)\left(g_\omega^2(t)^{-1}\alpha_0\right)dt \end{aligned} \tag{3.14}$$

or, equivalently,

$$d\tilde{\alpha}_\omega(t) = d\left(g_\omega^2(t)^{-1}\alpha_0\right) \tag{3.15}$$

$$\overset{(3.13)}{=} T_eR_{g^2_\omega(t)^{-1}}\left[\sum_{j=1}^{k_2}\left(-\left(H_j^2\alpha_0\right)\delta W^{2,j}_\omega(t)+\frac{1}{2}\left(\nabla_{H_j^2}H_j^2\right)\alpha_0 dt\right)-u(t)\alpha_0 dt\right]$$

Since we assume U^* to be a finite dimensional vector space, the difference between the Stratonovich and Itô integrals (see (2.1)) yields

$$\sum_{j=1}^{k_2}\left(H_j^2\left(g^2_\omega(t)^{-1}\alpha_0\right)\right)\delta W^{2,j}_\omega(t)$$
$$=\sum_{j=1}^{k_2}\left(\left(H_j^2\left(g^2_\omega(t)^{-1}\alpha_0\right)\right)dW^{2,j}_\omega(t)+\frac{1}{2}d[\![H_j^2\left(g^2_\omega(\cdot)^{-1}\alpha_0\right),W^{2,j}_\omega]\!]_t\right).$$

By the same procedure as in (3.14), the (local) martingale part of $H_j^2(g^2_\omega(\cdot)^{-1}\alpha_0)$ is equal to $-\sum_{i=1}^{k_2}\int_0^\cdot H_j^2H_j^2(g^2_\omega(s)^{-1}\alpha_0)dW^{2,i}(s)$. Therefore, by (2.4) and (2.5) we derive

$$\sum_{j=1}^{k_2}d[\![H_j^2\left(g^2_\omega(\cdot)^{-1}\alpha_0\right),W^{2,j}_\omega]\!]_t=-\sum_{j=1}^{k_2}H_j^2(H_j^2(g^2_\omega(t)^{-1}\alpha_0))dt$$

Using (3.14) we have,

$$\text{(3.16)}\quad d\tilde\alpha_\omega(t)=-\sum_{j=1}^{k_2}\left(H_j^2(\tilde\alpha_\omega(t))dW^{2,j}_\omega(t)\right)$$
$$+\frac{1}{2}\sum_{j=1}^{k_2}H_j^2\left(H_j^2\left(\tilde\alpha_\omega(t)\right)\right)dt+\frac{1}{2}\sum_{j=1}^{k_2}\left(\nabla_{H_j^2}H_j^2\right)\left(\tilde\alpha_\omega(t)\right)dt-u(t)\left(\tilde\alpha_\omega(t)\right)dt,$$

or, equivalently,

$$\text{(3.17)}\qquad d\tilde\alpha_\omega(t)=T_eR_{g^2_\omega(t)^{-1}}\left[\sum_{j=1}^{k_2}-\left(H_j^2\alpha_0\right)dW^{2,j}_\omega(t)\right]$$
$$+T_eR_{g^2_\omega(t)^{-1}}\left[\sum_{j=1}^{k_2}\left(\frac{1}{2}H_j^2(H_j^2\alpha_0)dt+\frac{1}{2}\left(\nabla_{H_j^2}H_j^2\right)\alpha_0 dt\right)-u(t)\alpha_0 dt\right]$$

Note that (3.14), respectively (3.16), are equations for $\tilde\alpha_\omega(t)=\left(g^2_\omega(t)\right)^{-1}\alpha_0$, a new variable that is introduced by the reduction process. On the other hand, (3.15) and its Itô version (3.17) are equations for the right logarithmic derivative of $g^2_\omega(t)$ with known right hand side (once $u(t)$ is determined). Thus, if one wants to solve for $g^2_\omega(t)$, this second group of equations are more convenient, even though they do not use the reduced variables.

Taking the expectation in (3.10), as Itô integrals have vanishing expectation values, we have

$$\frac{d}{dt}\mathbb{E}\left[\tilde{\alpha}_\omega(t)\right] = \frac{1}{2}\sum_{j=1}^{k_2}\mathbb{E}\left[H_j^2\left(H_j^2\left(\tilde{\alpha}_\omega(t)\right)\right)\right] - \mathbb{E}\left[\tilde{u}^2(t)(\tilde{\alpha}_\omega(t))\right], \tag{3.18}$$

where $\tilde{u}^2(t)$ is defined in (3.9). Since the Lie algebra representation on U^* commutes with integration over Ω (as linear maps), we deduce that $\alpha(t) := \mathbb{E}\left[g_\omega^2(t)^{-1}\alpha_0\right] \in U^*$ satisfies the equation

$$\frac{d}{dt}\alpha(t) = \frac{1}{2}\sum_{j=1}^{k_2} H_j^2\left(H_j^2\alpha(t)\right) - \tilde{u}^2(t)\alpha(t). \tag{3.19}$$

Step 2. We prove the first equation in (3.8). Recall from (3.4) that, for every (deterministic) curve $\mathfrak{g}(\cdot) \in C^1([0,1];T_eG)$ satisfying $\mathfrak{g}(0) = \mathfrak{g}(T) = 0$ and $\varepsilon \in [0,1)$, $e_{\varepsilon,\mathfrak{g}}(\cdot) \in C^1([0,T];G)$ uniquely solves the following differential equation on G

$$\frac{d}{dt}e_{\varepsilon,\mathfrak{g}}(t) = \varepsilon T_e L_{e_{\varepsilon,\mathfrak{g}}(t)}\dot{\mathfrak{g}}(t), \qquad e_{\varepsilon,\mathfrak{g}}(0) = e.$$

Recall also the notation $g^i_{\omega,\varepsilon,\mathfrak{g}}(t) := g^i_\omega(t)e_{\varepsilon,\mathfrak{g}}(t)$, $t \in [0,T]$, $i = 1,2$.

By [3, Lemma 3.1], we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e_{\varepsilon,\mathfrak{g}}(t) = \mathfrak{g}(t), \qquad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e_{\varepsilon,\mathfrak{g}}(t)^{-1} = -\mathfrak{g}(t) \tag{3.20}$$

Since this computation is important in the proof, for the sake of completeness, we recall it below. Denoting by $\frac{D}{Dt}$ and $\frac{D}{D\epsilon}$ the covariant derivatives, induced by ∇ on G, along curves parametrized by t and ε, respectively, since the torsion vanishes, the Gauss Lemma yields

$$\begin{aligned}\frac{D}{Dt}\frac{d}{d\varepsilon}e_{\varepsilon,\mathfrak{g}}(t) &= \frac{D}{D\varepsilon}\frac{d}{dt}e_{\varepsilon,\mathfrak{g}}(t) = \frac{D}{D\varepsilon}\big(\varepsilon T_e L_{e_{\varepsilon,\mathfrak{g}}(t)}\dot{\mathfrak{g}}(t)\big)\\ &= T_e L_{e_{\varepsilon,\mathfrak{g}}(t)}\dot{\mathfrak{g}}(t) + \varepsilon\frac{D}{D\varepsilon}\big(T_e L_{e_{\varepsilon,\mathfrak{g}}(t)}\dot{\mathfrak{g}}(t)\big)\end{aligned} \tag{3.21}$$

Taking $\varepsilon = 0$, since $e_{0,\mathfrak{g}}(t) = e$ for all t, we obtain $\frac{D}{Dt}\frac{d}{d\varepsilon}\big|_{\varepsilon=0}e_{\varepsilon,\mathfrak{g}}(t) = \dot{\mathfrak{g}}(t)$. However, $t \mapsto \frac{d}{d\varepsilon}\big|_{\varepsilon=0}e_{\varepsilon,\mathfrak{g}}(t)$ is a curve in the vector space T_eG and hence $\frac{d}{dt}\frac{d}{d\varepsilon}\big|_{\varepsilon=0}e_{\varepsilon,\mathfrak{g}}(t) = \frac{D}{Dt}\frac{d}{d\varepsilon}\big|_{\varepsilon=0}e_{\varepsilon,\mathfrak{g}}(t) = \dot{\mathfrak{g}}(t)$. The first equality in (3.20) is then a consequence of $\mathfrak{g}(0) = 0$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}e_{\varepsilon,\mathfrak{g}}(0) = 0$. Finally, since

$$\frac{d}{d\varepsilon}e_{\varepsilon,\mathfrak{g}}(t)^{-1} = -T_e R_{e^{-1}_{\varepsilon,\mathfrak{g}}(t)} T_{\varepsilon,\mathfrak{g}} L_{e^{-1}_{\varepsilon,\mathfrak{g}}(t)}\frac{d}{d\varepsilon}e_{\varepsilon,\mathfrak{g}}(t),$$

the second equality in (3.20) follows from the first.

Next, we need the following identity from the proof of [3, Theorem 3.2]

$$
(3.22)\qquad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(T_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}L_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)^{-1}}\frac{\mathscr{D}^\nabla g^1_{\omega,\varepsilon,\mathcal{g}}(t)}{dt}\right)
= \frac{d\mathcal{g}(t)}{dt}+\operatorname{ad}_{\tilde{u}^1(t)}\mathcal{g}(t)-\frac{1}{2}\sum_{j=1}^{k_1}\left(\nabla_{\operatorname{ad}_{\mathcal{g}(t)}H^1_j}H^1_j+\nabla_{H^1_j}\left(\operatorname{ad}_{\mathcal{g}(t)}H^1_j\right)\right),
$$

where $\tilde{u}^1(t)$ is given by (3.9). To prove it, introduce $H^{1,\varepsilon}_j(t) := \operatorname{Ad}_{e^{-1}_{\varepsilon,\mathcal{g}}(t)}H^1_j$, use (3.7), Itô's formula, and recall that $e_{\varepsilon,\mathcal{g}}(t)$ is of bounded variation (the time derivative of a product obeys the usual Leibniz formula only if at least one of the factors is of bounded variation). We get

$$
\begin{aligned}
dg^1_{\omega,\varepsilon,\mathcal{g}}(t) &= \sum_{j=1}^{k_1}T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}H^{1,\varepsilon}_j(t)\delta W^{1,j}_\omega(t)+T_eL_{g_{\omega,\varepsilon,\mathcal{g}}(t)}\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathcal{g}}(t)}\left(u(t)-\frac{1}{2}\sum_{j=1}^{k_1}\nabla_{H^1_j}H^1_j\right)dt\\
&\quad+T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}T_eL_{e_{\varepsilon,\mathcal{g}}(t)^{-1}}\dot{e}_{\varepsilon,\mathcal{g}}(t)dt\\
&\overset{(3.9)}{=}\sum_{j=1}^{k_1}T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}H^{1,\varepsilon}_j(t)\delta W^{1,j}_\omega(t)+T_eL_{g_{\omega,\varepsilon,\mathcal{g}}(t)}\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathcal{g}}(t)}\tilde{u}^1(t)dt\\
&\quad+T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}T_eL_{e_{\varepsilon,\mathcal{g}}(t)^{-1}}\dot{e}_{\varepsilon,\mathcal{g}}(t)dt.
\end{aligned}
$$

or, written in the Itô form,

$$
\begin{aligned}
dg^1_{\omega,\varepsilon,\mathcal{g}}(t) = T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(t)}&\left(\sum_{j=1}^{k_1}H^{1,\varepsilon}_j(t)dW^{1,j}_\omega(t)\right.\\
&\left.+\left(\frac{1}{2}\sum_{j=1}^{k_1}\nabla_{H^{1,\varepsilon}_j(t)}H^{1,\varepsilon}_j(t)+\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathcal{g}}(t)}\tilde{u}^1(t)+T_eL_{e_{\varepsilon,\mathcal{g}}(t)^{-1}}\dot{e}_{\varepsilon,\mathcal{g}}(t)\right)dt\right).
\end{aligned}
$$

By Itô's formula (for finite dimensional manifolds) and by (3.4), for every function $f \in C^2(G)$, the process N^f defined as follows is a real-valued local martingale,

$$
\begin{aligned}
N^f_\omega(t) :=& f(g^1_{\omega,\varepsilon,\mathcal{g}}(t))-f(g^1_{\omega,\varepsilon,\mathcal{g}}(0))\\
&-\frac{1}{2}\sum_{j=1}^{k_1}\int_0^t\left(T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(s)}H^{1,\varepsilon}_j(s)\big(T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(s)}H^{1,\varepsilon}_j(s)f\big)\right)(g^1_{\omega,\varepsilon,\mathcal{g}}(s))ds\\
&-\int_0^t\left(T_eL_{g^1_{\omega,\varepsilon,\mathcal{g}}(s)}\big(\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathcal{g}}(s)}\tilde{u}^1(s)+\varepsilon\dot{\mathcal{g}}(s)\big)f\right)(\xi^1_{\varepsilon,\mathcal{g}}(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&= f(g^1_{\omega,\varepsilon,\mathscr{g}}(t)) - f(g^1_{\omega,\varepsilon\mathscr{g}}(0)) \\
&\quad - \frac{1}{2}\sum_{j=1}^{k_1}\int_0^t \mathrm{Hess} f(g^1_\omega(s))\big(T_eL_{g^1_{\omega,\varepsilon,\mathscr{g}}(s)}H_j^{1,\varepsilon}(s), T_eL_{g^1_{\omega,\varepsilon,\mathscr{g}}(s)}H_j^{1,\varepsilon}(s)\big)ds \\
&\quad - \frac{1}{2}\sum_{j=1}^{k_1}\int_0^t T_eL_{g^1_{\omega,\varepsilon,\mathscr{g}}(s)}\nabla_{H_j^{1,\varepsilon}(s)}H_j^{1,\varepsilon}(s)f(g^1_{\omega,\varepsilon,\mathscr{g}}(s))ds \\
&\quad - \int_0^t\Big(T_eL_{g^1_{\omega,\varepsilon,\mathscr{g}}(s)}\big(\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathscr{g}}(s)}\tilde{u}^1(s) + \varepsilon\dot{\mathscr{g}}(s)\big)f\Big)(g^1_{\omega,\varepsilon,\mathscr{g}}(s))ds.
\end{aligned}
$$

Thus, formula (2.11) for the ∇-generalized derivative of the semimartingale $(\omega,t)\mapsto g^1_{\omega(t),\varepsilon,\mathscr{g}}(t)$ is applicable and we obtain

$$
T_{g^1_{\omega,\varepsilon,\mathscr{g}}(t)}L_{g^1_{\omega,\varepsilon,\mathscr{g}}(t)^{-1}}\frac{\mathscr{D}^\nabla g^1_{\omega,\varepsilon,\mathscr{g}}(t)}{dt} = \frac{1}{2}\sum_{j=1}^{k_1}\nabla_{H_j^{1,\varepsilon}(t)}H_j^{1,\varepsilon}(t) + \operatorname{Ad}_{e^{-1}_{\varepsilon,\mathscr{g}}(t)}\tilde{u}^1(t) + \varepsilon\dot{\mathscr{g}}(t).
$$

To prove (3.22), we take $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}$ of both sides. The third term yields $\frac{d}{dt}\mathscr{g}(t)$. To compute this derivative of the second term, use (3.20) and $e_{\varepsilon,\mathscr{g}}(0)=e$ to conclude

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\operatorname{Ad}_{e^{-1}_{\varepsilon,\mathscr{g}}(t)}\tilde{u}^1(t) = -\operatorname{ad}_{\mathscr{g}(t)}\tilde{u}^1(t) = \operatorname{ad}_{\tilde{u}^1(t)}\mathscr{g}(t)
$$

Finally, to compute this derivative of the first term, note that $H_j^{1,0}(t) = H_j^1$ for all t and that, again by (3.20), $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}H_j^{1,\varepsilon}(t) = -\operatorname{ad}_{\mathscr{g}(t)}H_j^1$, which yields

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\nabla_{H_j^{1,\varepsilon}(t)}H_j^{1,\varepsilon}(t) = -\nabla_{\operatorname{ad}_{\mathscr{g}(t)}H_j^1}H_j^1 - \nabla_{H_j^1}(\operatorname{ad}_{\mathscr{g}(t)}H_j^1)
$$

thus proving (3.22).

Since $g^i_{\omega,\varepsilon,\mathscr{g}}(t) := g^i_\omega(t)e_{\varepsilon,\mathscr{g}}(t)$ and $e_{0,\mathscr{g}}(t)=e$ for all $t\in[0,T]$, we conclude $g^i_{\omega,0,\mathscr{g}}(t) = g^i_\omega(t)$, for all $t\in[0,T]$, $i=1,2$. Therefore,

$$
\begin{aligned}
(3.23)\qquad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^2_{\omega,\varepsilon,\mathscr{g}}(t)^{-1}\alpha_0 &= -g^2_\omega(t)^{-1}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^2_{\omega,\varepsilon,\mathscr{g}}(t)\right)g^2_\omega(t)^{-1}\alpha_0 \\
&= -\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e_{\varepsilon,\mathscr{g}}(t)\right)g^2_\omega(t)^{-1}\alpha_0 \overset{(3.20)}{=} -\mathscr{g}(t)g^2_\omega(t)^{-1}\alpha_0.
\end{aligned}
$$

With (3.22) and (3.23), we can compute the ε-derivative of $J^{\nabla,\alpha_0,l}\left(g^1_{\omega,\varepsilon,\mathscr{g}}(\cdot), g^2_{\omega,\varepsilon,\mathscr{g}}(\cdot)\right)$ at $\varepsilon=0$. We get from (3.2)

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J^{\nabla,\alpha_0,l}\left(g^1_{\omega,\varepsilon,\mathscr{g}}(\cdot), g^2_{\omega,\varepsilon,\mathscr{g}}(\cdot)\right)
$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \left\langle \frac{\delta l}{\delta u}(t),\, \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_{g^1_{\omega,\varepsilon,\mathscr{g}}(t)} L_{g^1_{\omega,\varepsilon,\mathscr{g}}(t)^{-1}} \frac{\mathscr{D}^\nabla g^1_{\omega,\varepsilon,\mathscr{g}}(t)}{dt} \right\rangle \\
&\qquad + \mathbb{E}\int_0^T \left\langle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbb{E}\left[g^2_{\omega,\varepsilon,\mathscr{g}}(t)^{-1}\alpha_0 \right], \frac{\delta l}{\delta \alpha}(t) \right\rangle dt \\
&= \int_0^T \left\langle \frac{\delta l}{\delta u}(t),\, \frac{d\mathscr{g}(t)}{dt} + \mathrm{ad}_{\tilde{u}^1(t)}\mathscr{g}(t) - \frac{1}{2}\sum_{j=1}^{k_1} \left(\nabla_{\mathrm{ad}_{\mathscr{g}(t)} H^1_j} H^1_j + \nabla_{H^1_j}\big(\mathrm{ad}_{\mathscr{g}(t)} H^1_j\big)\right) \right\rangle dt \\
&\qquad - \int_0^T \left\langle \mathscr{g}(t)\left(\mathbb{E}\left[g^2_\omega(t)^{-1}\alpha_0\right]\right), \frac{\delta l}{\delta \alpha}(t) \right\rangle dt \\
&= \int_0^T \left\langle -\frac{d}{dt}\frac{\delta l}{\delta u}(t) + \mathrm{ad}^*_{\tilde{u}^1(t)} \frac{\delta l}{\delta u}(t) + \frac{\delta l}{\delta \alpha}(t) \diamond \alpha(t) + K\left(\frac{\delta l}{\delta u}(t)\right), \mathscr{g}(t) \right\rangle dt,
\end{aligned}
$$

where the last equality is obtained integrating by parts the first summand (the boundary terms vanish since $\mathscr{g}(0) = \mathscr{g}(T) = 0$), and applying the definitions of ad^*, $\diamond$ (see (3.1)), and K (see (3.10)).

Since $\mathscr{g} \in C^1([0,T]; T_eG)$ is arbitrary, $(g^1_\omega(\cdot), g^2_\omega(\cdot))$ is a critical point of $J^{\nabla,\alpha_0,l}$ if and only if $u(\cdot) \in C^1([0,T]; T_eG)$ satisfies the second equation in (3.8).

This proves statement (i).

(ii)The expressions of the variations (3.12) have been already found in the previous computations. Using them in the variational principle (3.11), we obtain

$$
\begin{aligned}
\delta \int_0^T l(u(t),\alpha(t))dt &= \int_0^T \left(\left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle + \left\langle \delta\alpha, \frac{\delta l}{\delta \alpha} \right\rangle \right) dt \\
&\overset{(3.12)}{=} \int_0^T \left(\left\langle \frac{\delta l}{\delta u}, \dot{v} + \mathrm{ad}_{\tilde{u}^1} v + K^*(v) \right\rangle - \left\langle v\alpha, \frac{\delta l}{\delta \alpha} \right\rangle \right) dt \\
&= \int_0^T \left\langle -\frac{d}{dt}\left(\frac{\delta l}{\delta u}\right) + \mathrm{ad}^*_{\tilde{u}^1} \frac{\delta l}{\delta u} + K\left(\frac{\delta l}{\delta u}\right) + \frac{\delta l}{\delta \alpha} \diamond \alpha, v \right\rangle dt.
\end{aligned}
$$

This vanishes for all smooth curves $v(t) \in T_eG$ vanishing at the endpoints if and only if the first equation in (3.8) holds. $\square$

Remark 3.3. As we shall see later, the conclusion of Theorem 3.2 still holds when G is the diffeomorphism group of a torus and the action of G on U^* is the pull back map.

If G is a finite dimensional Lie group and U a finite dimensional vector space, then $\tilde{\alpha}(\cdot)$ and $\alpha(\cdot)$ defined by (3.3) satisfy some SDE and ODE respectively. However, when G is the diffeomorphism group, as will be seen in the next section, $\tilde{\alpha}(\cdot)$ and $\alpha(\cdot)$ satisfy some SPDE and PDE, respectively. $\Diamond$

Instead of (3.7), let us consider semimartingales $h^m_\omega(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, defined

by

$$
(3.24) \qquad \begin{cases} dh^m_\omega(t) = T_e L_{h^m_\omega(t)} \left(\sum_{j=1}^{k_m} \left(H^m_j \delta W^{m,j}_\omega(t) \right) + u(t)dt \right), \\ h^m_\omega(0) = e, \end{cases}
$$

for $u(\cdot) \in C^1([0,T]; T_eG)$. Accordingly, define

$$
(3.25) \qquad \alpha(t) := \mathbb{E}\big[\tilde{\alpha}_\omega(t)\big] \in U^*, \qquad \tilde{\alpha}_\omega(t) := h^2_\omega(t)^{-1}\alpha_0.
$$

Then, proceeding as in the proof of Theorem 3.2, we get the following result.

Theorem 3.4. *Under the conditions of Theorem 3.2, suppose that the semimartingales $h^m_\omega(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, have the form* (3.24) *and their deformations are of the form* (3.6)*.*

(i) *Then $(h^1_\omega(\cdot), h^2_\omega(\cdot))$ is a critical point of $J^{\nabla,\alpha_0,l}$ (see* (3.2)*) if and only if the (non-random) curve $u(\cdot) \in C^1([0,T]; T_eG)$ coupled with $\alpha(\cdot) \in C^1([0,T]; U^*)$ (defined by* (3.25)*) satisfies the following* semidirect product Euler-Poincaré equation for stochastic particle paths:

$$
(3.26) \qquad \begin{cases} \frac{d}{dt}\frac{\delta l}{\delta u} = \mathrm{ad}^*_{u^1}\frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha + K\left(\frac{\delta l}{\delta u}\right), \\ \frac{d}{dt}\alpha(t) = \frac{1}{2}\sum_{j=1}^{k_2} H^2_j \left(H^2_j \alpha(t) \right) - u^2(t)\alpha(t), \end{cases}
$$

(ii) *The first equation in* (3.20) *is equivalent to the* dissipative Euler-Poincaré variational principle

$$
(3.27) \qquad \delta \int_0^T l(u(t), \alpha(t))dt = 0
$$

on $T_eG \times U^*$ defined by $l : T_eG \times U^* \to \mathbb{R}$, for variations of the form

$$
(3.28) \qquad \begin{cases} \delta u = \dot{v} + [u^1, v] + K^*(v) = \dot{v} + [u, v] + K^*(v), \\ \delta \alpha = -v\alpha \end{cases}
$$

where $v(t)$ is an arbitrary curve in T_eG vanishing at the endpoints, i.e., $v(0) = 0$, $v(T) = 0$.

Right invariant version. Due to relative sign changes in the equations of motion and the dissipative constrained variational principle, with a view to applications for the spatial representation in continuum mechanics, we give below the right invariant version of Theorem 3.2.

Let G be a Lie group endowed with a right invariant covariant derivative ∇, i.e., $\nabla_{v_1^R} v_2^R$ is a right invariant vector field, for any $v_1, v_2 \in T_eG$, where $v_i^R \in \mathfrak{X}(G)$ is

the right invariant vector field defined by $v_i^R(g) := T_e R_g v_i$, $i = 1, 2$. Recall that $[v_1, v_2]^R = -\left[v_1^R, v_2^R\right]$, where on the left hand side the bracket is in $\mathfrak{g}$, hence defined via left invariant vector fields (the standard left Lie algebra of G), and on the right, it is the usual Jacobi-Lie bracket of smooth vector fields on G.

Suppose that G acts on the right on a vector space U (and we will write the action of $g \in G$ on $u \in U$ by ug and similarly for the induced infinitesimal $\mathfrak{g}$-representation). In the procedure leading to Theorem 3.2 and its proof, we interchange all left actions and left translation operators by their right counterparts. Thus the action functional $J^{\nabla,\alpha_0,l}$ is defined in this right invariant case by

$$J^{\nabla,\alpha_0,l}(g_\omega^1(\cdot), g_\omega^2(\cdot)) := \mathbb{E}\int_0^T l\Big(T_{g_\omega^1(t)} R_{g_\omega^1(t)^{-1}} \frac{\mathscr{D}^\nabla g_\omega^1(t)}{dt}, \alpha(t)\Big) dt,\ g_\omega^1, g_\omega^2 \in \mathscr{S}(G), \tag{3.29}$$

where $T_{g_\omega^1(t)} R_{g_\omega^1(t)^{-1}}$ denotes the differential of the inverse of right translation $R_{g_\omega^1(t)^{-1}}$ at the point $g_\omega^1(t) \in G$ and

$$\alpha(t) := \mathbb{E}\left[\tilde{\alpha}_\omega(t)\right] \in U^*, \quad \tilde{\alpha}_\omega(t) := \alpha_0 g_\omega^2(t)^{-1}. \tag{3.30}$$

As before, for every (deterministic) curve $\mathscr{g}(\cdot) \in C^1([0,1]; T_e G)$ satisfying $\mathscr{g}(0) = \mathscr{g}(T) = 0$ and $\varepsilon \in [0, 1)$, let $e_{\varepsilon,\mathscr{g}}(\cdot) \in C^1([0,T]; G)$ be the unique solution of the (deterministic) time-dependent differential equation on G

$$\begin{cases} \frac{d}{dt} e_{\varepsilon,\mathscr{g}}(t) = \varepsilon T_e R_{e_{\varepsilon,\mathscr{g}}(t)} \dot{\mathscr{g}}(t), \\ e_{\varepsilon,\mathscr{g}}(0) = e. \end{cases} \tag{3.31}$$

Define

$$g^i_{\omega,\varepsilon,\mathscr{g}}(t) := e_{\varepsilon,\mathscr{g}}(t) g^i_\omega(t);$$

we can consider (right invariant) critical points of $J^{\nabla,\alpha_0,l}$ as before. Moreover, formulas (3.7)–(3.19) still hold with all the left translations and left $\mathfrak{g}$-actions replaced by right ones. For example, the right version of the time evolution (3.19) of $\alpha(t)$ (given this time by (3.30)) takes the following form for right invariant linear connections on G and right G-representations

$$\frac{d}{dt}\alpha(t) = \frac{1}{2}\sum_{j=1}^{k_2} \left(\alpha(t) H_j^2\right) H_j^2 - \alpha(t)\tilde{u}^2(t), \tag{3.32}$$

while the definition (3.9) of $\tilde{u}^2(t)$ remains unchanged. The semimartingale $g_\omega^2(\cdot) \in \mathscr{S}(G)$ is the solution of the following two equivalent stochastic differential equations (in Stratonovich and Itô form)

$$d\tilde{\alpha}_\omega(t) = d\left(\alpha_0 g_\omega^2(t)^{-1}\right) \tag{3.33}$$

$$
\begin{aligned}
&= T_e L_{g^2_\omega(t)^{-1}} \left[\sum_{j=1}^{k_2} \left(-\left(\alpha_0 H^2_j\right) \delta W^{2,j}_\omega(t) + \frac{1}{2}\alpha_0 \left(\nabla_{H^2_j} H^2_j \right) dt \right) - \alpha_0 u(t) dt \right] \\
&= T_e L_{g^2_\omega(t)^{-1}} \left[\sum_{j=1}^{k_2} - \left(\alpha_0 H^2_j\right) dW^{2,j}_\omega(t) \right] \\
&\qquad + T_e L_{g^2_\omega(t)^{-1}} \left[\sum_{j=1}^{k_2} \left(\frac{1}{2}(\alpha_0 H^2_j) H^2_j dt + \frac{1}{2}\alpha_0 \left(\nabla_{H^2_j} H^2_j \right) dt \right) - \alpha_0 u(t) dt \right].
\end{aligned}
$$

Finally, in the proof of the analogue of (3.22) (on the right hand side the second and third summand change sign), one uses $H^{1,\varepsilon}_j(t) := \operatorname{Ad}_{e_{\varepsilon,g}(t)} H^1_j$.

The right invariant version of Theorem 3.2 is the following.

Theorem 3.5. *Assume G is a finite dimensional Lie group endowed with a right invariant linear connection ∇. Let U be a finite dimensional right G-representation space. Suppose that the semimartingales $g^i_\omega(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, have the following form,*

$$
\left\{
\begin{array}{l}
dg^m_\omega(t) = T_e R_{g^m_\omega(t)} \Big(\sum_{j=1}^{k_m} \left(H^m_j \delta W^{m,j}_\omega(t) - \frac{1}{2} \nabla_{H^m_j} H^m_j dt \right) + u(t) dt \Big), \\
g^m_\omega(0) = e,
\end{array}
\right. \tag{3.34}
$$

and that their deformations are given by (3.6).

(i) *Then $(g^1_\omega(\cdot), g^2_\omega(\cdot))$ is a critical point of $J^{\nabla,\alpha_0,l}$ given by* (3.29) *if and only if the (non-random) curve $u(\cdot) \in C^1([0,T]; T_e G)$ coupled with $\alpha(\cdot) \in C^1([0,T]; U^*)$ (defined by* (3.30)*) satisfies the following* semidirect product Euler-Poincaré equation for stochastic particle paths:

$$
\left\{
\begin{array}{l}
\frac{d}{dt} \frac{\delta l}{\delta u} = -\operatorname{ad}^*_{\tilde{u}^1} \frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha - K\left(\frac{\delta l}{\delta u} \right), \\
\frac{d}{dt} \alpha(t) = \frac{1}{2} \sum_{j=1}^{k_2} \left(\alpha(t) H^2_j \right) H^2_j - \alpha(t) \tilde{u}^2(t)
\end{array}
\right. \tag{3.35}
$$

where $\alpha(t)$, $\tilde{u}^1$, and K are defined by (3.30), (3.9), *and* (3.10), *respectively.*

(ii) *The first equation in* (3.35) *is equivalent to the* dissipative Euler-Poincaré variational principle

$$
\delta \int_0^T l(u(t), \alpha(t)) dt = 0 \tag{3.36}
$$

on $T_e G \times U^*$ defined by $l : T_e G \times U^* \to \mathbb{R}$, for variations of the form

$$
\left\{
\begin{array}{l}
\delta u = \dot{v} - [\tilde{u}^1, v] - K^*(v) = \dot{v} + [u, v] - \dfrac{1}{2} \left[\displaystyle\sum_{j=1}^{k_1} \nabla_{H^1_j} H^1_j, v \right] - K^*(v), \\
\\
\delta \alpha = -\alpha v
\end{array}
\right. \tag{3.37}
$$

where $v(t)$ is an arbitrary curve in T_eG vanishing at the endpoints, i.e., $v(0) = 0$, $v(T) = 0$.

Remark 3.6. Under some special conditions, the operator $K(u)$ takes a familiar form. More precisely, if ∇ is the (right invariant) Levi-Civita connection and if we assume that $\nabla_{H_i} H_i = 0$ for each i, we have,

$$K(u) = -\frac{1}{2}\sum_i \big(\nabla_{H_i}\nabla_{H_i} u + \mathbf{R}(u, H_i)H_i\big), \quad \forall u \in \mathfrak{g}, \tag{3.38}$$

where $\mathbf{R}$ is the Riemannian curvature tensor with respect to ∇. In particular, if $\{H_i\}$ is an orthonormal basis of $\mathfrak{g}$, then $K(u) = -\frac{1}{2}\big(\Delta u + \mathrm{Ric}(u)\big)$, where $\Delta u := \Delta U(x)|_{g=e}$ for the right invariant vector fields $U(g) := T_e R_g u$, $\forall u \in \mathfrak{g}$, $g \in G$.

A proof can be found in [3], Proposition 3.5. $\Diamond$

The right invariant version of Theorem 3.5 reads:

Theorem 3.7. *Assume G is a finite dimensional Lie group endowed with a right invariant linear connection ∇. Let U be a finite dimensional right G-representation space. Suppose that the semimartingales $h^i_\omega(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$, have the following form,*

$$\begin{cases} dh^m_\omega(t) = T_e R_{h^m_\omega(t)}\Big(\sum_{j=1}^{k_m}\big(H^m_j \delta W^{m,j}_\omega(t)\big) + u(t)dt\Big), \\ h^m_\omega(0) = e, \end{cases} \tag{3.39}$$

and that their deformations are given by (3.6).

(i) *Then $(h^1_\omega(\cdot), h^2_\omega(\cdot))$ is a critical point of $J^{\nabla,\alpha_0,l}$ given by* (3.29) *if and only if the* (*non-random*) *curve $u(\cdot) \in C^1([0,T]; T_eG)$ coupled with $\alpha(\cdot) \in C^1([0,T]; U^*)$* (*defined by* (3.30)) *satisfies the following* semidirect product Euler-Poincaré equation for stochastic particle paths:

$$\begin{cases} \frac{d}{dt}\frac{\delta l}{\delta u} = -\mathrm{ad}^*_{u^1}\frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha - K\left(\frac{\delta l}{\delta u}\right), \\ \frac{d}{dt}\alpha(t) = \frac{1}{2}\sum_{j=1}^{k_2}\big(\alpha(t)H^2_j\big) H^2_j - \alpha(t)u^2(t) \end{cases} \tag{3.40}$$

where K is defined by (3.10) *and $\alpha(t)$ is defined by*

$$\alpha(t) := \mathbb{E}\left[\tilde{\alpha}_\omega(t)\right] \in U^*, \quad \tilde{\alpha}_\omega(t) := \alpha_0 h^2_\omega(t)^{-1}. \tag{3.41}$$

(ii) *The first equation in* (3.40) *is equivalent to the* dissipative Euler-Poincaré variational principle

$$\delta \int_0^T l(u(t), \alpha(t))dt = 0 \tag{3.42}$$

on $T_eG \times U^*$ defined by $l : T_eG \times U^* \to \mathbb{R}$, for variations of the form

$$\begin{cases} \delta u = \dot{v} - [u^1, v] - K^*(v) = \dot{v} + [u, v] - K^*(v), \\ \delta\alpha = -\alpha v \end{cases} \tag{3.43}$$

where $v(t)$ is an arbitrary curve in T_eG vanishing at the endpoints, i.e., $v(0) = 0$, $v(T) = 0$.

4 Application to the MHD equation

We begin by recalling from [20] and [61] the necessary standard facts about the group of diffeomorphisms on a smooth compact boundaryless n-dimensional manifold M. Then, when we present the compressible MHD equations, we shall specialize to the periodic case, i.e., we shall take $M = \mathbb{T}^3$, the usual three dimensional flat torus. Of course, all of the discussion below can be easily extended to smooth manifolds without boundary.

Let M be a smooth compact boundaryless n-dimensional manifold. Define

$$G^s := \left\{ g : M \to M \text{ is a bijection} \mid g, g^{-1} \in H^s(M, M) \right\},$$

where $H^s(M, M)$ denotes the manifold of Sobolev maps of class $s > 1 + \frac{n}{2}$ from M to itself. The condition $s > \frac{n}{2}$ suffices to ensure the manifold structure of $H^s(M, M)$; only for such regularity class does the notion of an H^s-map from M to itself make intrinsic sense. If $s > 1 + \frac{n}{2}$ (the additional regularity is needed in order to ensure that all elements of G^s are C^1 and hence the inverse function theorem is applicable), then G^s is an open subset in $H^s(M, M)$, so it is a C^∞ Hilbert manifold. Moreover it is a group under composition between maps, right translation by any element is smooth, left translation and inversion are only continuous, and G^s is a topological group (relative to the underlying manifold topology) (see [20], [61]); thus, G^s is not a Lie group. Since G^s is an open subset of $H^s(M, M)$, the tangent space T_eG^s to the identity $e : M \to M$ coincides with the Hilbert space $\mathfrak{X}^s(M)$ of H^s vector fields on M. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. The failure of G^s to be a Lie group is mirrored by the fact that $\mathfrak{X}^s(M)$ is not a Lie algebra: the usual Jacobi-Lie bracket of vector fields, i.e., $[X, Y][f] = X[Y[f]] - Y[X[f]]$ for any $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, where $X[f] := \mathbf{d}f(X)$ is the differential of f in the direction X, loses a derivative for finite differentiability class of vector fields and thus $[\cdot, \cdot] : \mathfrak{X}^s(M) \times \mathfrak{X}^s(M) \to \mathfrak{X}^{s-1}(M)$ is not an operation on $\mathfrak{X}^s(M)$. In general, the tangent space $T_\eta G^s$ at an arbitrary $\eta \in G^s$ is $T_\eta G^s = \left\{ U : M \to TM \text{ of class } H^s \mid U(m) \in T_{\eta(m)}M \right\}$. If $s > 1 + \frac{n}{2}$ and $X \in \mathfrak{X}^s(M)$, then its global (since M is compact) flow $\mathbb{R} \ni t \mapsto F_t \in G^s$ exists and is a C^1-curve in G^s (see, e.g., [61, Theorem 2.4.2]). The candidate of what should have been the Lie group exponential map is $\exp : T_eG^s = \mathfrak{X}^s(M) \ni X \mapsto F_1 \in G^s$, where F_t is the flow of X; however, exp does not cover a neighborhood of the identity and it is not

C^1. Therefore, all classical proofs in the theory of finite dimensional Lie groups based on the exponential map, break down for G^s. From now on, we shall always assume $s > 1 + \frac{n}{2}$.

Since right translation is smooth, each $X \in \mathfrak{X}(M)$ induces a C^∞ right invariant vector field $X^R \in \mathfrak{X}(G^s)$ on G^s, defined by $X^R(\eta) := X \circ \eta$. With this notation, we have the identity $\left[X^R, Y^R\right](e) = [X, Y]$, for any $X, Y \in \mathfrak{X}(M)$. This is the analogue of saying that $\mathfrak{X}(M)$ is the "right Lie algebra" of G^s.

Assume, in addition, that M is connected, oriented, and Riemannian; denote by $\langle \cdot, \cdot \rangle$ the Riemannian metric. Let μ_g be the Riemannian volume form on M, whose expression in local coordinates $(x^1, \ldots, x^n)$ is $\mu_g = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$, where $g_{ij} := \langle \partial/\partial x^i, \partial/\partial x^j \rangle$ for all $i, j = 1, \ldots n$. Let $K : T(TM) \to TM$ be the connector of the Levi-Civita connection (with Christoffel symbols Γ^i_{jk}) defined by the Riemannian metric on M; in local coordinates, this intrinsic object, which is a vector bundle map $K : T(TM) \to TM$ covering the canonical vector bundle projection $\tau_M : TM \to M$, has the expression $K(x^i, u^i, v^i, w^i) = \left(x^i, w^i + \Gamma^i_{jk} u^j v^k\right)$. The connector K satisfies hence the identity $\tau_M \circ K = \tau_M \circ \tau_{TM}$ and has the property that the vector bundles $\tau_{TM} : T(TM) \to TM$ and $\ker K \oplus \ker T\tau_M \to TM$ are isomorphic. These two properties characterize the connector. Conversely, the connector K determines ∇ in the following manner: $\nabla_X Y := K \circ TY \circ X$ for any $X, Y \in \mathfrak{X}(M)$.

The Riemannian structure on M induces the *weak* L^2, or *hydrodynamic*, *metric* $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\eta$ on G^s given by

$$\langle\!\langle U_\eta, V_\eta \rangle\!\rangle_\eta := \int_M \langle U_\eta(m), V_\eta(m) \rangle_{\eta(m)} \, d\mu_g(m),$$

for any $\eta \in G^s$, $U_\eta, V_\eta \in T_\eta G^s$. This means that the association $\eta \mapsto \langle\!\langle \cdot, \cdot \rangle\!\rangle_\eta$ from G^s to the vector bundle of symmetric covariant two-tensors on G^s is smooth but that for every $\eta \in G^s$, the map $T_\eta G^s \ni U_\eta \mapsto \langle\!\langle U_\eta, \cdot \rangle\!\rangle \in T^*_\eta G^s$, where $T^*_\eta G^s$ denotes the linear continuous functionals on TG^s, is only injective and not, in general, surjective. This weak metric is not right invariant (because of the Jacobian appearing in the change of variables formula in the integral).

The usual proof for finite dimensional Lie groups showing the existence of a unique Levi-Civita connection associated to a Riemannian metric breaks down, because $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is weak; the proof would only show uniqueness. However, $K^0 : T(TG^s) \to TG^s$ given by $K^0(\mathcal{Z}_{U_\eta}) := K \circ \mathcal{Z}_{U_\eta}$, where $\mathcal{Z}_{U_\eta} \in T_{U_\eta}(TG^s)$, is a connector for the vector bundle $\tau_{G^s} : TG^s \to G^s$ (since $\tau_{G^s} \circ K^0 = \tau_{G^s} \circ \tau_{TG^s}$ and the vector bundles $\tau_{TG^s} : T(TG^s) \to TG^s$ and $\ker K^0 \oplus \ker T\tau_{G^s} \to TG^s$ are isomorphic). Here, $\mathcal{Z}_{U_\eta} \in T_{U_\eta}(TG^s)$ means that $\mathcal{Z}_{U_\eta} : M \to T(TM)$ satisfies $\tau_{TM}\left(\mathcal{Z}_{U_\eta}(m)\right) \in T_{\eta(m)}M$. The covariant derivative ∇^0 on G^s is defined by $\nabla^0_{\mathcal{X}} \mathcal{Y} := K^0 \circ T\mathcal{Y} \circ \mathcal{X}$, for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(G^s)$. This is the Levi-Civita connection associated to the weak metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ since it is torsion free ($\nabla^0_{\mathcal{X}} \mathcal{Y} - \nabla^0_{\mathcal{Y}} \mathcal{X} = [\mathcal{X}, \mathcal{Y}]$, for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}^s(M)$, where $[\mathcal{X}, \mathcal{Y}]$ is the Jacobi-Lie bracket of vector fields on G^s) and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$-compatible ($\mathcal{Z}[\langle\!\langle \mathcal{X}, \mathcal{Y} \rangle\!\rangle] = \langle\!\langle \nabla_{\mathcal{Z}} \mathcal{X}, \mathcal{Y} \rangle\!\rangle + \langle\!\langle \mathcal{X}, \nabla_{\mathcal{Z}} \mathcal{Y} \rangle\!\rangle$, for

all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathfrak{X}^s(M)$); see [20, Theorem 9.1]. Uniqueness of such a connection follows from the weak non-degeneracy of the metric $\langle\langle \cdot, \cdot \rangle\rangle$. There is an explicit formula for right-invariant covariant derivatives on diffeomorphism groups (see, e.g., [27, page 6]). For ∇^0 this formula is

$$\left(\nabla^0_{\mathcal{X}} \mathcal{Y}\right)(\eta) := \frac{\partial}{\partial t}\left(\mathcal{Y}(\eta_t) \circ \eta_t^{-1}\right) \circ \eta + \left(\nabla_{X^\eta} Y^\eta\right) \circ \eta,$$

where $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(G^s)$, $X^\eta := \mathcal{X} \circ \eta^{-1}, Y^\eta := \mathcal{Y} \circ \eta^{-1} \in \mathfrak{X}^s(M)$, and $t \mapsto \eta_t$ is a C^1 curve in G^s such that $\eta_0 = \eta$ and $\frac{d}{dt}\Big|_{t=0} \eta_t = \mathcal{X}(\eta)$; this formula is identical to [65, (3.1)]. Note that each term on the right hand side of this formula is only of class H^{s-1} and, nevertheless, their sum is of class H^s because of the abstract definition of the covariant derivative on G^s. A similar phenomenon occurs with the geodesic spray $TG^s \to T(TG^s)$ of the weak Riemannian metric; its existence and smoothness for boundaryless M was proved in [20], i.e., the geodesic spray is in $\mathfrak{X}(TG^s)$. If M has a boundary, this statement is false.

The discussion above shows that one cannot apply the theorems of Section 3 to the infinite dimensional group G^s directly. For infinite dimensional problems, they serve only as a guideline and direct proofs are needed, which is what we do below. However, for each important formula, we shall point out the analogue in the finite dimensional abstract setting of Section 3 which inspired the result and we show that Theorem 3.4 still holds for the model presented here.

From now on we consider the case $M = \mathbb{T}^3$. We focus on the following type of SDEs on G^s,

$$\begin{cases} dg_\omega(t,\theta) = \sum_{j=1}^k H_j(g_\omega(t,\theta))\delta W^j_\omega(t) + \tilde{u}(t, g_\omega(t,\theta))dt \\ g_\omega(0,\theta) = \theta, \quad \theta \in \mathbb{T}^3, \end{cases}$$

where $H_j \in \mathfrak{X}(G^s)$, $\tilde{u}(t) := u(t) - \frac{1}{2}\sum_{j=1}^k \nabla^0_{H_j} H_j$, $u \in C^1([0,T]; T_e G^s)$ is non-random, and $dg_\omega(t,\theta)$ is the Itô differential of $g_\omega(t,\theta)$ relative to the time variable t. In particular, take the constant vector fields $H_1 = \sqrt{2\nu}(1,0,0)$, $H_2 = \sqrt{2\nu}(0,1,0)$, $H_3 = \sqrt{2\nu}(0,0,1)$ on $\mathbb{T}^3$, where $\nu \geq 0$ is a constant. This is understood in the trivialization $T\mathbb{T}^3 = \mathbb{T}^3 \times \mathbb{R}^3$, so $H_1, H_2, H_3 : \mathbb{T}^3 \to \mathbb{R}^3$ are constant maps. Define the process $g^\nu_\omega(t,\theta)$ by

$$\begin{cases} dg^\nu_\omega(t,\theta) = \sqrt{2\nu} dW_\omega(t) + u(t, g^\nu_\omega(t,\theta))dt \\ g^\nu_\omega(0,\theta) = \theta. \end{cases} \tag{4.1}$$

where $W_\omega(t)$ is a $\mathbb{R}^3$-valued Brownian motion and $dW_\omega(t)$ is the Itô differential of $W_\omega(t)$ with respect to the time variable.

By standard theory on stochastic flows (see, e.g., [51]), if u is regular enough, i.e., $u \in C^1([0,T]; T_e G^{s'})$ for some $s' > 0$ large enough, then $g^\nu_\omega(t,\cdot) \in G^s$ for every $t \in [0,T]$. From now on, for simplicity, we always assume u to be regular enough. As in

[45, Section 6], let U^* be some linear space which can be a space of functions, densities, or differential forms on $\mathbb{T}^3$. The action of G^s on U^* is the pull back map and the action of "Lie algebra" T_eG^s on U^* is the Lie derivative. Let $\alpha_0 \in U^*$, $\tilde{\alpha}(t) := \alpha_0 g^\nu_\omega(t)^{-1}$, and $\alpha(t) := \mathbb{E}\big[\tilde{\alpha}(t)\big]$. We first study the equation satisfied by $\alpha(t)$, given some specific choice of $\alpha_0 \in U^*$.

If we take $\alpha_0 = A_0(\theta)\cdot \mathbf{d}\theta := \sum_{i=1}^3 A_{0,i}(\theta)\mathbf{d}\theta_i$ to be a C^2 one-form on $\mathbb{T}^3$, we derive the following result (see also an equivalent expression in [24], equations (32)-(34)). Formula (4.4) below is the analogue of (3.32), derived here by hand for the infinite dimensional group G^s.

Proposition 4.1. *Let $g^\nu_\omega(t)$ be given by* (4.1)*. Define*

$$\tilde{\alpha}_\omega(t,\theta) := \left(\alpha_0 g^\nu_\omega(t,\cdot)^{-1}\right)(\theta) = \left(\left(g^\nu_\omega(t,\cdot)^{-1}\right)^* \alpha_0\right)(\theta) := \tilde{A}_\omega(t,\theta)\cdot \mathbf{d}\theta := \sum_{i=1}^3 \tilde{A}_i(t,\theta,\omega)\mathbf{d}\theta_i,$$

where $\left(g^\nu_\omega(t,\cdot)^{-1}\right)^*$ denotes the pull back map by $g^\nu_\omega(t,\cdot)^{-1}$, and

$$\alpha(t,\theta) := \mathbb{E}[\tilde{\alpha}_\omega(t,\theta)] := A(t,\theta)\cdot \mathbf{d}\theta := \sum_{i=1}^3 A_i(t,\theta)\mathbf{d}\theta_i.$$

Then $\tilde{A}$ satisfies the following SPDE,

$$\begin{aligned} d\tilde{A}_i(t,\theta,\omega) =& -\sum_{j=1}^3 \sqrt{2\nu}\partial_j \tilde{A}_i(t,\theta,\omega)dW^j_\omega(t) \\ &-\sum_{j=1}^3 \Big(u_j(t,\theta)\partial_j\tilde{A}_i(t,\theta,\omega) \\ &+\tilde{A}_j(t,\theta,\omega)\partial_i u_j(t,\theta)\Big)dt + \nu\Delta\tilde{A}_i(t,\theta,\omega)dt,\ i=1,2,3, \end{aligned} \tag{4.2}$$

where we use the notation $u(t) := (u_1(t), u_2(t), u_3(t))$ and ∂_j and Δ stand for the partial derivative and the Laplacian with respect to the space variable θ of $\tilde{A}_\omega(t,\theta)$ respectively. Equation (4.2) *can also be expressed as*

$$\begin{aligned} d\tilde{A}_\omega(t,\theta) =& -\sqrt{2\nu}\nabla\tilde{A}_\omega(t,\theta)\cdot dW_\omega(t) \\ &-\Big(u(t,\theta)\times \mathrm{curl}\tilde{A}_\omega(t,\theta) - \nabla(u(t,\theta)\cdot\tilde{A}_\omega(t,\theta))\Big)dt + \nu\Delta\tilde{A}_\omega(t)dt \end{aligned} \tag{4.3}$$

(the term $d\tilde{A}_\omega(t,\theta)$ above denotes the Itô differential of $\tilde{A}_\omega(t,\theta)$ with respect to the time variable) and we have

$$\begin{cases} \partial_t A(t,\theta) = \big(u(t,\theta)\times \mathrm{curl}A(t,\theta) - \nabla(u(t,\theta)\cdot A(t,\theta))\big) + \nu\Delta A(t,\theta), \\ A(0,\theta) = A_0(\theta). \end{cases} \tag{4.4}$$

Proof. We use the methods in [18, Lemma 4.1] and [18, Proposition 4.2]. It is not hard to see that, for the C^2 (note that we assume u to be regular) spatial process $\tilde{A}_i(t,\theta,\omega)$, there exist adapted spatial processes $h_{ij}(t,\theta,\omega)$ and $z_i(t,\theta,\omega)$, $1 \leqslant i,j \leqslant 3$, such that,

$$d\tilde{A}_i(t,\theta,\omega) = \sum_{j=1}^{3} h_{ij}(t,\theta,\omega) dW_\omega^j(t) + z_i(t,\theta,\omega)dt, \quad i = 1,2,3. \tag{4.5}$$

We compute below the expressions of $h_{ij}(t,\theta,\omega)$ and $z_i(t,\theta,\omega)$.

Notice that by the definition of $\tilde{\alpha}_\omega(t,\theta)$, $(g_\omega^\nu(t,\theta))^* \tilde{\alpha}_\omega(t,\theta) = \alpha(0,\theta)$ is a constant with respect to the time variable. Since

$$(g_\omega^\nu(t,\theta))^* \tilde{\alpha}_\omega(t,\theta) = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} \tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega) V_{ij}(t,\theta,\omega)\right) d\theta_j,$$

where the process $V_{ij}(t,\theta,\omega) := \partial_j g_i^\nu(t,\theta,\omega)$, we get for each $1 \leq j \leq 3$,

$$d\Big(\sum_{i=1}^{3} \tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega) V_{ij}(t,\theta,\omega)\Big) = 0, \tag{4.6}$$

By (4.5) and the generalized Itô formula for spatial processes, [51, Theorem 3.3.1.], we get

$$\begin{aligned} d\tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega) &= \sum_{j=1}^{3}\Big(h_{ij}(t, g_\omega^\nu(t,\theta),\omega) + \sqrt{2\nu}\partial_j \tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega)\Big) dW_\omega^j(t) \\ &+ \Bigg(z_i(t, g_\omega^\nu(t,\theta),\omega) + \nu\Delta\tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega) \\ &\quad + \sum_{j=1}^{3}\Big(u_j\left(t, g_\omega^\nu(t,\theta)\right)\partial_j \tilde{A}_i(t, g_\omega^\nu(t,\theta),\omega) + \sqrt{2\nu}\partial_j h_{ij}\left(t, g_\omega^\nu(t,\theta),\omega\right)\Big)\Bigg) dt. \end{aligned} \tag{4.7}$$

Using the SDE (4.1) and the theory of the stochastic flows in [51, Theorem 3.3.3.], we have,

$$dV_{ij}(t,\theta,\omega) = \sum_{k=1}^{3} \partial_k u_i(t, g_\omega^\nu(t,\theta)) V_{kj}(t,\theta,\omega) dt. \tag{4.8}$$

In particular, the martingale part of the above equality vanishes due to the fact that the diffusion coefficients in (4.1) are constant. According to (4.7) and (4.8), for each

$1 \leq j \leq 3$, the time Itô differential is

$$
\begin{aligned}
d&\left(\sum_{i=1}^{3} \tilde{A}_i(t, g^\nu_\omega(t,\theta), \omega) V_{ij}(t,\theta,\omega)\right)\\
&= \sum_{i,k=1}^{3} \Big(h_{ik}(t, g^\nu_\omega(t,\theta),\omega) + \sqrt{2\nu}\partial_k \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega)\Big) V_{ij}(t,\theta,\omega) dW^k_\omega(t)\\
&\quad + \sum_{i=1}^{3}\left(\sum_{k=1}^{3}\Big(u_k(t, g^\nu_\omega(t,\theta))\partial_k \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega) + \sqrt{2\nu}\partial_k h_{ik}(t, g^\nu_\omega(t,\theta),\omega)\Big)\right.\\
&\qquad\qquad \left. + z_i(t, g^\nu_\omega(t,\theta),\omega) + \nu\Delta \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega)\right) V_{ij}(t,\theta,\omega) dt\\
&\quad + \sum_{i,k=1}^{3} \tilde{A}_k(t, g^\nu_\omega(t,\theta),\omega)\partial_i u_k(t, g^\nu_\omega(t,\theta)) V_{ij}(t,\theta,\omega) dt
\end{aligned}
$$

Hence from (4.6), we derive for each $1 \leq j, m \leq 3$,

$$
\sum_{i=1}^{3} \Big(h_{im}(t, g^\nu_\omega(t,\theta),\omega) + \sqrt{2\nu}\partial_m \tilde{A}_i(t, g^\nu_\omega(t,\theta))\Big) V_{ij}(t,\theta,\omega) = 0 \tag{4.9}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{3} &\left(z_i(t, g^\nu_\omega(t,\theta)) + \nu\Delta \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega)\right.\\
&\quad + \sum_{k=1}^{3}\Big(u_k(t, g^\nu_\omega(t,\theta))\partial_k \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega) + \sqrt{2\nu}\partial_k h_{ik}(t, g^\nu_\omega(t),\omega)\\
&\qquad \left. + \tilde{A}_k(t, g^\nu_\omega(t,\theta),\omega)\partial_i u_k(t, g^\nu_\omega(t,\theta))\Big)\right) V_{ij}(t,\theta,\omega) = 0.
\end{aligned} \tag{4.10}
$$

Since $\{V_{ij}(t,\theta,\omega)\}_{1\leqslant i,j\leqslant 3}$ is a non-degenerate matrix-valued process (see, e.g., [51]), from (4.9) we deduce that, for each $1 \leq i, j \leq 3$,

$$
h_{ij}(t, g^\nu_\omega(t,\theta),\omega) = -\sqrt{2\nu}\partial_j \tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega).
$$

Noticing that, ω-almost surely, $\theta \mapsto g^\nu_\omega(t,\theta)$ is a diffeomorphism for each fixed t, we get

$$
h_{ij}(t,\theta,\omega) = -\sqrt{2\nu}\partial_j \tilde{A}_i(t,\theta,\omega),\ \forall \theta \in \mathbb{T}^3, \tag{4.11}
$$

which is the expression for $h_{ij}(t,\theta,\omega)$.

Since $\{V_{ij}(t,\theta,\omega)\}_{1\leqslant i,j\leqslant 3}$ is non-degenerate, by (4.10), for each $1 \le i \le 3$,

$$z_i(t, g^\nu_\omega(t,\theta), \omega) = -\nu\Delta\tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega) - \sum_{k=1}^{3}\Big(u_k(t, g^\nu_\omega(t,\theta))\partial_k\tilde{A}_i(t, g^\nu_\omega(t,\theta),\omega)) \\ -\sqrt{2\nu}\partial_k h_{ik}(t, g^\nu_\omega(t,\theta),\omega) + \tilde{A}_k(t, g^\nu_\omega(t,\theta),\omega)\partial_i u_k(t, g^\nu_\omega(t,\theta))\Big).$$

Taking (4.11) into the above equation and due to the property that, ω-almost surely, $\theta \mapsto g_\omega(t,\theta)$ is a diffeomorphism for each fixed t, we obtain the expression for $z_i(t,\theta,\omega)$, namely,

$$\begin{aligned} z_i(t,\theta,\omega) =&\nu\Delta\tilde{A}_i(t,\theta,\omega) \\ &-\sum_{k=1}^{3}\Big(u_k(t,\theta)\partial_k\tilde{A}_i(t,\theta,\omega) + \tilde{A}_k(t,\theta,\omega)\partial_i u_k(t,\theta)\Big), \quad \forall\theta \in \mathbb{T}^3. \end{aligned} \tag{4.12}$$

Combining (4.5), (4.11), and (4.12) proves (4.2). We can check that (4.2) is equivalent to (4.3) by direct computation. Taking the expectation of the two sides of equation (4.3) and recalling that $u(t)$ is non-random, (4.4) follows. $\square$

Remark 4.2. In Proposition 4.1, U^* is taken to be a space of differential forms on $\mathbb{T}^3$. Note that the action of G^s on U^* is the pull back map and the action of "Lie algebra" T_eG^s on U^* is the Lie derivative. Then, for $H_1 = \sqrt{2\nu}(1,0,0)$, $H_2 = \sqrt{2\nu}(0,1,0)$, $H_3 = \sqrt{2\nu}(0,0,1)$ and $\alpha(t,\theta) = A(t,\theta)\cdot \mathbf{d}\theta$ we have

$$\nabla_{H_j}H_j = 0, \quad \sum_{j=1}^{3}\alpha(t)H_jH_j = \nu\Delta A(t,\theta)\cdot\mathbf{d}\theta,$$
$$\alpha(t)u(t) = \big(u(t,\theta)\times \mathrm{curl}\ A(t,\theta) - \nabla(u(t,\theta)\cdot A(t,\theta))\big)\cdot\mathbf{d}\theta,$$

which implies that the second equation in (3.35) still holds. $\Diamond$

Remark 4.3. By the same procedure as in the proof of Proposition 4.1, if α_0 is replaced by another term, such as a function or a density, we can still prove the corresponding evolution equation for $\alpha(t,\theta) := \mathbb{E}\big[\big(\alpha_0 g^\nu_\omega(t,\cdot)\big)^{-1}\big)(\theta)\big]$.

For example, if $\alpha_0 = \beta_0 : \mathbb{T}^3 \to \mathbb{R}$ is a C^2 function, then $\alpha(t,\theta)$ satisfies the following transport equation, see [18],

$$\begin{cases} \partial_t\alpha(t,\theta) = -u(t,\theta)\cdot\nabla\alpha(t,\theta) + \nu\Delta\alpha(t,\theta), \\ \alpha(0,\theta) = \beta_0(x). \end{cases}$$

If $\alpha_0 = D_0(\theta)d^3\theta$ is a density (volume form), write $\alpha(t,\theta) = D(t,\theta)d^3\theta$. Then $D(t,\theta)$ satisfies the following forward Kolmogorov equation (or Fokker-Plank equation),

$$\begin{cases} \partial_t D(t,\theta) = -\nabla\cdot(Du)(t,\theta) + \nu\Delta D(t,\theta), \\ D(0,\theta) = D_0(\theta), \end{cases} \tag{4.13}$$

where $\nabla \cdot$ is the divergence operator. Moreover, if $\alpha_0 = D_0(\theta)d^3\theta$ is a probability measure, let $\tilde{g}^\nu_\omega(t,\theta)$ be the process satisfying

$$d\tilde{g}^\nu_\omega(t,\theta) = \sqrt{2\nu}dW_\omega(t) + u(t, \tilde{g}^\nu_\omega(t,\theta))dt$$

whose initial distribution $\tilde{g}^\nu_\omega(0,\theta)$ is $D_0(\theta)d^3\theta$. Suppose that for every $t \in [0,T]$, the distribution of $\tilde{g}^\nu_\omega(t,\theta)$ is of the form $D(t,\theta)d^3\theta$: then $D(t,\theta)$ satisfies (4.13). $\Diamond$

By the analysis in [3, Section 4.2], for the (infinite dimensional group) G^s, and using the right-invariant version of definition (2.11), we have

$$T_{g^\nu_\omega(t,\theta)}R_{g^\nu_\omega(t,\theta)^{-1}}\frac{\mathscr{D}^{\nabla^0}g^\nu_\omega(t,\theta)}{dt} = u(t,\theta).$$

Let $\alpha_0 := (b_0(x), \mathbf{B}_0(\theta)\cdot\mathbf{dS}, D_0(\theta)d^3\theta)$, where b_0 is a C^2 function on $\mathbb{T}^3$, $\mathbf{B}_0(\theta)\cdot\mathbf{dS}$ is an exact two-form on $\mathbb{T}^3$, i.e., there is some one-form $A_0(\theta)\cdot\mathbf{d}\theta$ such that

$$\mathbf{B}_0(\theta)\cdot\mathbf{dS} = \mathbf{d}\Big(A_0(\theta)\cdot\mathbf{d}\theta\Big) = \sum_{1\le j<k\le 3, i\ne j, i\ne k}\big(\mathrm{curl}A_0(\theta)\big)_i\mathbf{d}\theta_j\wedge\mathbf{d}\theta_k, \tag{4.14}$$

and $D_0(\theta)d^3\theta$ is a density on $\mathbb{T}^3$. We let U^* denote the vector space of all such triples $(b_0(x), \mathbf{B}_0(\theta)\cdot\mathbf{dS}, D_0(\theta)d^3\theta)$.

As in [45, equation (7.16)], let $l : T_eG^s\times U^*\to\mathbb{R}$ be defined by

$$l(u,b,\mathbf{B},D) = \int_{\mathbb{T}^3}\left(\frac{D(\theta)}{2}|u(\theta)|^2 - D(\theta)e(D(\theta),b(\theta)) - \frac{1}{2}|\mathbf{B}(\theta)|^2\right)d^3\theta,$$

where $u\in T_eG^s = \mathfrak{X}^s(\mathbb{T}^3)$ is the Eulerian (spatial) velocity of the fluid, $b\in C^2(\mathbb{T}^3)$ is the entropy function, $\mathbf{B}(\theta)\cdot\mathbf{dS}$ is an exact differential two-form as in (4.14) representing the magnetic field in the fluid, $D(\theta)d^3\theta$ is a density on $\mathbb{T}^3$ representing the mass density of the fluid, and the function $e(D,b)$ is the fluid's specific internal energy. The pressure $p(D,b)$ and the temperature $T(D,b)$ of the fluid are given in terms of a thermodynamic equation of state for the specific internal energy e, namely $\mathbf{d}e = -p\mathbf{d}\big(\frac{1}{D}\big) + T\mathbf{d}b = D^2\frac{\partial e}{\partial D}\mathbf{d}D + T\mathbf{d}b$. It is assumed that $c^2 := \frac{\partial p}{\partial D} > 0$, where c is the adiabatic sound speed.

Now we proceed as in Section 3 deducing the relevant formulas by hand. As in (3.2) we define,

$$\mathbf{J}^{\nabla_0,\alpha_0,l}\big(g^{\mu_1}_\omega, g^{\mu_2}_\omega, g^{\mu_3}_\omega, g^{\mu_4}_\omega\big) := \mathbb{E}\int_0^T l\left(u_\omega(t), b(t), \mathbf{B}(t), D(t)\right)dt, \tag{4.15}$$

where the stochastic processes $g^{\mu_i}_\omega$, $1\le i\le 4$, are defined by (4.1) with $\mu_i\ge 0$ and the same $u(t,\theta)$, and the curve $t\mapsto(u_\omega(t), b(t), \mathbf{B}(t), D(t))$ in U^* is defined by

$$\left\{\begin{array}{l} u_\omega(t,\theta) := T_{g^{\mu_1}_\omega(t,\theta)}R_{g^{\mu_1}_\omega(t,\theta)^{-1}}\frac{\mathscr{D}^{\nabla^0}g^{\mu_1}_\omega(t,\theta)}{dt}, \\ b(t,\theta) := \mathbb{E}\left[\left((g^{\mu_2}_\omega(t,\cdot)^{-1})^* b_0\right)(\theta)\right], \\ \mathbf{B}(t,\theta)\cdot\mathbf{dS} := \mathbb{E}\left[\left((g^{\mu_3}_\omega(t,\cdot)^{-1})^*(\mathbf{B}_0(\theta)\cdot\mathbf{dS})\right)(\theta)\right], \\ D(t,\theta)d^3\theta := \mathbb{E}\left[\left((g^{\mu_4}_\omega(t,\cdot)^{-1})^*(D_0(\theta)d^3\theta)\right)(\theta)\right], \end{array}\right. \tag{4.16}$$

where $\left(g_\omega^{\mu_i}(t,\cdot)^{-1}\right)^*$ denotes the pull back map by $g_\omega^{\mu_i}(t,\cdot)^{-1} : \mathbb{T}^3 \to \mathbb{T}^3$.

For each non-random $v \in C^1([0,1]; T_eG^s)$ with $v(0) = v(T) = 0$ which is regular enough, the perturbation (3.4) (for right invariant systems) in the formulation here is determined by the following flow $e_{\varepsilon,v}(t,\cdot) \in G^s$, see e.g., [3], [17],

$$\begin{cases} \frac{de_{\varepsilon,v}(t,\theta)}{dt} = \varepsilon \frac{\partial}{\partial t} v(t, e_{\varepsilon,v}(t,\theta)) \\ e_{\varepsilon,v}(0,\theta) = \theta \end{cases}$$

We say that $(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4})$ is a critical point of $\mathbf{J}^{\nabla^0,\alpha_0,l}$ if for each $v \in C^1([0,T]; T_eG^s)$ with $v(0) = v(T) = 0$,

$$\left.\frac{d\mathbf{J}^{\nabla^0,\alpha_0,l}\left(g_{\varepsilon,v}^{\mu_1}, g_{\varepsilon,v}^{\mu_2}, g_{\varepsilon,v}^{\mu_3}, g_{\varepsilon,v}^{\mu_4}\right)}{d\varepsilon}\right|_{\varepsilon=0} = 0,$$

where $g_{\varepsilon,v}^{\mu_i}(t,\omega)(\theta) := e_{\varepsilon,v}(t, g_\omega^{\mu_i}(t,\theta))$.

Then we have the following result.

Theorem 4.4. *The semimartingale $(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4})$ is a critical point of $\mathbf{J}^{\nabla^0,\alpha_0,l}$ if and only if the following equations hold for $(u(t), b(t), \mathbf{B}(t), D(t))$,*

$$\begin{cases} \partial_t u + u\cdot\nabla u + \frac{\nabla p}{D} + \frac{\mathbf{B}\times \mathrm{curl}\mathbf{B}}{D} = \mu_1 \Delta u + (\mu_1 - \mu_4)\frac{u\Delta D}{D} + 2\mu_1 \langle \nabla \log D, \nabla u\rangle \\ \partial_t b = -u\cdot\nabla b + \mu_2 \Delta b \\ \partial_t \mathbf{B} = \mathrm{curl}(u\times \mathbf{B}) + \mu_3\Delta\mathbf{B} \\ \partial_t D = -\nabla\cdot(Du) + \mu_4 \Delta D \\ \nabla\cdot\mathbf{B} = 0, \end{cases} \tag{4.17}$$

where p is the pressure term.

Proof. By the analysis in [3, Section 4.2], properties (3.20) and (3.22) still hold for the diffeomorphism group G^s. Since the right action of G^s here is the pull back map, it is not different to check the validity of (3.23) for our present formulation. Therefore we can repeat the argument to show that $(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4})$ is a critical point of $\mathbf{J}^{\nabla^0,\alpha_0,l}$ if and only if the following equation holds,

$$\frac{d}{dt}\frac{\delta l}{\delta u} = -\mathrm{ad}^*_{u(t)}\frac{\delta l}{\delta u} + \frac{\delta l}{\delta b}\diamond b(t) + \frac{\delta l}{\delta \mathbf{B}}\diamond \mathbf{B}(t) + \frac{\delta l}{\delta D}\diamond D(t) - K\Big(\frac{\delta l}{\delta u}\Big), \tag{4.18}$$

where $b(t), \mathbf{B}(t), D(t)$ are defined by (4.16), and the operator $K : T_eG^s \to T_eG^s$ is defined by (3.10), and where we also use the property $\nabla^0_{H_i} H_i = 0$; hence $\tilde{u}(t) = u(t)$.

Note that $\mathbf{B}_0(\theta)\cdot\mathbf{dS}=\mathbf{d}\big(A_0(\theta)\cdot\mathbf{d}\theta\big)$ for some one-form $A_0(\theta)\cdot\mathbf{d}\theta$, and

$$\begin{aligned}\mathbf{B}(t,\theta)\cdot\mathbf{dS}:&=\mathbb{E}\Big[\big(\big(g_\omega^{\mu_3}(t,.)^{-1}\big)^*\big(\mathbf{B}_0(\theta)\cdot b\mathbf{dS}\big)\big)(\theta)\Big]\\&=\mathbb{E}\Big[\big(\big(g_\omega^{\mu_3}(t,.)^{-1}\big)^*\mathbf{d}\big(A_0(\theta)\cdot\mathbf{d}\theta\big)\big)(\theta)\Big]\\&=\mathbb{E}\Big[\mathbf{d}\big(\big(g_\omega^{\mu_3}(t,.)^{-1}\big)^*\big(A_0(\theta)\cdot\mathbf{d}\theta\big)\big)(\theta)\Big]\\&=\mathbf{d}\big(A(t,\theta)\cdot\mathbf{d}\theta\big)(\theta),\end{aligned}$$

where

$$A(t,\theta)\cdot\mathbf{d}\theta:=\mathbb{E}\Big[\big(\big(g^{\mu_3}(t,.)^{-1}\big)^*\big(A_0(\theta)\cdot\mathbf{d}\theta\big)\big)(\theta)\Big],\quad \mathrm{curl}A(t)=\mathbf{B}(t).$$

By Proposition 4.1, equation (4.4) holds for $A(t)$ with viscosity constant $\nu=\mu_3$, and hence $\mathbf{B}(t)=\mathrm{curl}A(t)$ satisfies,

$$\partial_t\mathbf{B}=\mathrm{curl}(u\times\mathbf{B})+\mu_3\Delta\mathbf{B}.$$

We also have $\nabla\cdot\mathbf{B}(t)=\nabla\cdot(\mathrm{curl}A(t))=0$.

Moreover, from Remark 4.3, we know that $(b(t),\mathbf{B}(t),D(t))$ satisfies the last four equations in (4.17). Since $H_1=\sqrt{2\mu_1}(1,0,0)$, $H_2=\sqrt{2\mu_1}(0,1,0)$, $H_3=\sqrt{2\mu_1}(0,0,1)$, by the analysis in [3, Proposition 4.1], for every $v\in T_eG^s$,

$$\frac{1}{2}\sum_{j=1}^{3}\big(\nabla_{\mathrm{ad}_vH_j}H_j+\nabla_{H_j}(\mathrm{ad}_vH_j)\big)=-\mu_1\Delta v;$$

therefore we have $K\big(\frac{\delta l}{\delta u}\big)=-\mu_1\Delta\big(Du\big)$.

From the computation in [45, Section 7], particularly (7.4), (7.18) and (7.19), we get,

$$\begin{aligned}-\,\mathrm{ad}_u^*\frac{\delta l}{\delta u}+\frac{\delta l}{\delta b}\diamond b+\frac{\delta l}{\delta\mathbf{B}}\diamond\mathbf{B}+\frac{\delta l}{\delta D}\diamond D&\\=-Du\cdot\nabla u-u\nabla\cdot(Du)-\mathbf{B}\times\mathrm{curl}\mathbf{B}-\nabla p&\end{aligned}$$

for some function p, the pressure.

Combining all the above equalities into (4.18), we obtain,

$$\partial_t(Du)=-Du\cdot\nabla u-u\nabla\cdot(Du)-\mathbf{B}\times\mathrm{curl}\mathbf{B}-\nabla p+\mu_1\Delta(Du),\tag{4.19}$$

and, since

$$\partial_tD=-\nabla\cdot(Du)+\mu_4\Delta D,$$

we derive the first equation in (4.17). ☐

Remark 4.5. In the proof of the Theorem we see that the first equation in (3.35) holds (see (4.18)) and, by Remarks 4.2 and 4.3, the second equation is also true. Hence Theorem 3.5 still holds for our infinite dimensional model.

Remark 4.6. The equation satisfied by D is the analogue of the continuity equation for mass conservation. We have

$$\frac{d}{dt}\int D d\theta = \int [-\nabla \cdot (Du) + \mu_4 \Delta D] d\theta = 0.$$

Remark 4.7. The reason for choosing processes g^{μ_i} with different constants μ_i is that the viscosity constants in equation (4.17) are different.

Furthermore, if we take $\mu_1 = \mu_4 = \nu$ for some $\nu > 0$, we get,

$$\tag{4.20} \begin{cases} \partial_t u + u \cdot \nabla u + \frac{1}{D}\nabla p + \frac{1}{D}\mathbf{B} \times \mathrm{curl}\mathbf{B} = \nu \Delta u + 2\nu \langle \nabla \log D, \nabla u \rangle \\ \partial_t b = -u \cdot \nabla b + \mu_2 \Delta b \\ \partial_t \mathbf{B} = \mathrm{curl}(u \times \mathbf{B}) + \mu_3 \Delta \mathbf{B} \\ \partial_t D = -\nabla \cdot (Du) + \nu \Delta D \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

where u is the fluid velocity, b is the entropy, $\mathbf{B}$ is the magnetic field, p is the pressure, D is the mass density, T is the temperature, and ν, μ_3, and μ_2 are constants representing viscosity, resistivity sand diffusivity, respectively.

This is essentially the equation given in [70, Page 81], where ν is the coefficient of kinematic viscosity, assumed to be uniform (recall that the kinematic viscosity is the ratio of the shear viscosity to the density of the fluid). The term $2\nu\langle \nabla \log D, \nabla u\rangle$ does not appear there. $\diamondsuit$

Remark 4.8. In particular, if we take $D(t) = 1$, $b(t) = 1$ for every $t \in [0, T]$ in (4.20), we obtain the following viscous MHD equation, see e.g. [71],

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + \mathbf{B} \times \mathrm{curl}\mathbf{B} = \nu \Delta u \\ \partial_t \mathbf{B} = \mathrm{curl}(u \times \mathbf{B}) + \mu_3 \Delta \mathbf{B} \\ \nabla \cdot u = 0, \ \nabla \cdot \mathbf{B} = 0. \end{cases} \qquad \diamondsuit$$

Remark 4.9. The reason why the equation for the density (see (4.17) and (4.13)) contains the Laplacian term, and therefore is different from the deterministic case, is due to the fact the the underlying paths $g^\nu_\omega(t)$ are now random, with Brownian diffusion coefficient of intensity given by the viscosity. Thus we have, for all regular functions f on $\mathbb{T}^3$,

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{T}^3} f(g^\nu_\omega(t,\theta)) d^3\theta\right] = \mathbb{E}\left[\int_{\mathbb{T}^3} \left((u \cdot \nabla f) + \nu \Delta f\right)(g^\nu_\omega(t,\theta)) d^3\theta\right],$$

which gives precisely the Fokker-Planck equation for D. $\diamondsuit$

Remark 4.10. One can check that the energy decays along the solutions of the compressible equations (4.20). For this, the presence of the term $2\nu\langle\nabla \log D, \nabla u\rangle$ is important. $\Diamond$

Remark 4.11. The expression $\nabla \log D$ in the balance of linear momentum equation (the first equation in (4.20)) is reminiscent of Brenner's model ([7, 8]) for compressible fluids which uses two velocities: a mass velocity, that appears only in the continuity (mass conservation) equation, and a volume velocity (momentum per unit mass of fluid) derived from the motion of particles, that appears, together with the mass velocity, in both the balance of linear momentum and the energy conservation equations. The fundamental constitutive equation in Brenner's model (mass, linear momentum, and energy conservation) is the statement that the difference between the volume and mass velocities is proportional to $\nabla \log D$, thus leading to a system with only one velocity, but having additional terms which involve $\nabla \log D$. It should be emphasized that if the fluid is homogeneous and incompressible, the two velocities coincide. We refer to [25] for a detailed presentation and an in-depth mathematical analysis of Brenner's system.

While the system (4.20) involves a single velocity, the fluid part in (4.20) seems to indicate that the velocity u is Brenner's volume velocity and that some of the terms in his model may be due to stochastic motion of the fluid particles. It would be interesting to investigate the relationship of Brenner's model to the diffeomorphism group version of the semidirect product Euler-Poincaré equation for stochastic particle paths in Theorem 3.5. A different approach, where both Itô calculus with respect to forward and backward filtrations describing forward time evolution and its reversal are used, can be found in [50]. It also exhibits links to Brenner's model. $\Diamond$

References

[1] Abraham R., Marsden J.E. and Ratiu, T.,, *Manifolds, Tensor Analysis, and Applications*, Appl. Math. Sci. **75**, Springer-Verlag (1988).

[2] M. Arnaudon and A. B. Cruzeiro, Lagrangian Navier-Stokes diffusions on manifolds: variational principle and stability, *Bull. Sci. Math.*, **136**(8), 857-881 (2012).

[3] M. Arnaudon, X. Chen and A. B. Cruzeiro, Stochastic Euler-Poincaré reduction, *J. Math. Phys.*, **55**, 081507 (2014).

[4] J.-M. Bismut, *Mécanique Aléatoire*, Lecture Notes in Mathematics, **866**, Springer-Verlag, Berlin-New York, 1981.

[5] J.-M. Bismut, Mécanique aléatoire, in *Tenth Saint Flour Probability Summer School 1980* (Saint Flour, 1980), 1–100, Lecture Notes in Math., **929**, Springer, Berlin-New York, 1982.

[6] N. Bou-Rabee and H. Owhadi, Stochastic variational integrators, *IMA J. Numer. Anal.*, **29**, 421–443 (2008).

[7] H. Brenner, Kinematics of volume transport, *Physica A*, **349**, 11–59 (2005).

[8] H. Brenner, Navier-Stokes revisited, *Physica A*, **349**, 60–132 (2005).

[9] M. Bruveris, F. Gay-Balmaz, D. D. Holm, and T. S. Ratiu, The momentum map representation of images, *J. Nonlinear Sci.*, **21**(1), 115–150 (2011).

[10] M. Castrillón López, P. García Pérez, and T. S. Ratiu, Euler-Poincaré reduction on principal bundles, *Lett. Math. Phys.*, **58**(2), 167180 (2001).

[11] M. Castrillón López and T. S. Ratiu, Reduction in principal bundles: covariant Lagrange-Poincar equations, *Comm. Math. Phys.*, **236**(2), 223–250 (2003).

[12] M. Castrillón López, T. S. Ratiu, and S. Shkoller, Reduction in principal fiber bundles: covariant Euler-Poincaré equations, *Proc. Amer. Math. Soc.*, **128**(7), 2155–2164 (2000).

[13] H. Cendra, D. D. Holm, J. E. Marsden, and T.S. Ratiu, Lagrangian reduction, the Euler-Poincaré equations, and semidirect products, *Geometry of Differential Equations*, 1–25, Amer. Math. Soc. Transl. Ser. 2, **186**, Amer. Math. Soc., Providence, RI, 1998.

[14] H. Cendra, J. E. Marsden, S, Pekarsky, and T.S. Ratiu, Variational principles for Lie-Poisson and Hamilton-Poincaré equations, *Mosc. Math. J.*, **3**(3), 833–867, 1197–1198.

[15] H. Cendra, J. E. Marsden, and T.S. Ratiu, Lagrangian reduction by stages, *Mem. Amer. Math. Soc.*, **152**, no. 722, (2001).

[16] H. Cendra, J. E. Marsden, and T.S. Ratiu, Geometric mechanics, Lagrangian reduction, and nonholonomic systems, in *Mathematics Unlimited - 2001 and Beyond*, 221–273, Springer-Verlag, Berlin, 2001.

[17] F. Cipriano and A. B. Cruzeiro, Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus, *Comm. Math. Phys.*, **275**(1), 255–267 (2007).

[18] P. Constantin and G. Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations, *Comm. Pure Appl. Math.*, **61**(3) 330–345 (2008).

[19] F. Demoures, F. Gay-Balmaz, and T. S. Ratiu, Multisymplectic variational integrators for nonsmooth Lagrangian continuum mechanics, preprint.

[20] D. G. Ebin and J. E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.*, **92**, 102–163 (1970).

[21] D. Ellis, F. Gay-Balmaz, D. D. Holm, V. Putkaradze, and T. S. Ratiu, Symmetry reduced dynamics of charged molecular strands, *Arch. Ration. Mech. Anal.*, **197**(3), 811902, (2010).

[22] D. Ellis, F. Gay-Balmaz, D. D. Holm, and T. S. Ratiu, Lagrange-Poincaré field equations, *J. Geom. Phys.*, **61**(11), 2120–2146, (2011).

[23] M. Emery, *Stochastic Calculus in Manifolds*, Springer, Universitext, Springer-Verlag (1989).

[24] G. L. Eyink, Stochastic line motion and stochastic flux conservation for nonideal hydromagnetic models, *J. Math. Phys.*, **50**, 083102 (2009).

[25] E. Feireisl and A. Vasseur, New perspectives in fluid dynamics: mathematical analysis of a model proposed by Howard Brenner, in *New Directions in Mathematical Fluid Mechanics*, 153179, Birkhäuser Verlag, Basel.

[26] R. C. Fetecau, J. E. Marsden, M. Ortiz, and M. West, Nonsmooth Lagrangian mechanics and variational collision integrators, *SIAM J. Appl. Dyn. Syst.*, **2**(3), 381–416 (2003).

[27] F. Gay-Balmaz, *Infinite Dimensional Geodesic Flows and the Universal Teichmüller Space*, Ph.D. Thesis, Ecole Polytechnique Fédérale de Lausanne, November 21, 2008.

[28] F. Gay-Balmaz, D. D. Holm, D. Meier, T. S. Ratiu, and F.-X. Vialard, Invariant higher-order variational problems, *Comm. Math. Phys.*, **309**(2), 413–458 (2012).

[29] F. Gay-Balmaz, D. D. Holm, D. Meier, T. S. Ratiu, and F.-X. Vialard, Invariant higher-order variational problems II, *J. Nonlinear Sci.*, **22**(4), 553–597 (2012).

[30] F. Gay-Balmaz, D. D. Holm, V. Putkaradze, and T. S. Ratiu, Exact geometric theory of dendronized polymer dynamics, *Adv. in Appl. Math.*, **48**(4), 535–574 (2012).

[31] F. Gay-Balmaz, D. D. Holm, and T. S. Ratiu, Variational principles for spin systems and the Kirchhoff rod, *J. Geom. Mech.*, **1**(4), 417–444, (2009).

[32] F. Gay-Balmaz, D. D. Holm, and T. S. Ratiu, Higher order Lagrange-Poincaré and Hamilton-Poincaré reductions, *Bull. Braz. Math. Soc. (N.S.)*, **42**(4), 579–606 (2011).

[33] F. Gay-Balmaz, D. D. Holm, and T. S. Ratiu, Geometric dynamics of optimization, *Commun. Math. Sci.*, *11*(1), 163–231 (2013).

[34] F. Gay-Balmaz, J. E. Marsden, and T. S. Ratiu, Reduced variational formulations in free boundary continuum mechanics, *J. Nonlinear Sci.*, **22**(4), 463497 (2012).

[35] F. Gay-Balmaz, M. Monastyrsky, and T. S. Ratiu, Lagrangian reductions and integrable systems in condensed matter, *Comm. Math. Phys.*, **335**(2), 609–636 (2015).

[36] F. Gay-Balmaz and T. S. Ratiu, Reduced Lagrangian and Hamiltonian formulations of Euler-Yang-Mills fluids, *J. Sympl. Geom.*, **6**(2), 189–237 (2008).

[37] F. Gay-Balmaz and T. S. Ratiu, The geometric structure of complex fluids, *Adv. in Appl. Math.*, **42**(2), 176–275 (2009).

[38] F. Gay-Balmaz and T. S. Ratiu, A new Lagrangian dynamic reduction in field theory, *Ann. Inst. Fourier (Grenoble)*, **60**(3), 1125–1160 (2010).

[39] F. Gay-Balmaz and T. S. Ratiu, Geometry of nonabelian charged fluids, *Dyn. Partial Differ. Equ.*, **8**(1), 5–19, (2011).

[40] F. Gay-Balmaz and T. S. Ratiu, Clebsch optimal control formulation in mechanics, *J. Geom. Mech.*, **3**(1), 41–79, (2011).

[41] F. Gay-Balmaz, T. S. Ratiu, and C. Tronci, Euler-Poincaré approaches to nematodynamics, *Acta Appl. Math.*, **120**, 127151, (2012).

[42] F. Gay-Balmaz, T. S. Ratiu, and C. Tronci, Equivalent theories of liquid crystal dynamics, *Arch. Ration. Mech. Anal.*, **210**(3), 773–811 (2013).

[43] S. Hochgerner and T. S. Ratiu, Geometry of non-holonomic diffusion, *J. Eur. Math. Soc.*, **17**, 273-319 (2015), DOI 10.4171/JEMS/504.

[44] D. D. Holm, Variational principles for stochastic fluid dynamics, *Proc. R. Soc. A*, **471**: 20140963. http://dx.doi.org/10.1098/rspa.2014.0963

[45] D. D. Holm, J. E. Marsden, and T. S. Ratiu, The Euler-Poincaré equations and semidirect products with applications to continuum mechanics, *Adv. Math.*, **137**, 1–81 (1998).

[46] D. D. Holm, J. E. Marsden, and T. S. Ratiu, The Euler-Poincaré equations in geophysical fluid dynamics, in *Large-Scale Atmosphere-Ocean Dynamics*, Vol. II, 251–300, Cambridge Univ. Press, Cambridge, 2002.

[47] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Universitext, Springer-Verlag (1981).

[48] C. Kane, J. E. Marsden, M. Ortiz, and M. West, Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems, *Int. J. Numer. Methods Eng.*, **49**(10), 1295–1325 (2000).

[49] M. Kobilarov, J. E. Marsden, and G. S. Sukhatme, Geometric discretization of nonholonomic systems with symmetries, *Discrete Contin. Dyn. Syst. Ser. S*, **3**(1), 61–84 (2010).

[50] T. Koide and T. Kodama, Navier-Stokes, Gross-Pitaevskii and generalized diffusion equations using the stochastic variational method, *J. Phys. A*, **45**(25) 255204, 18 pp. (2012).

[51] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, North Holland Math. Lib. (1990).

[52] J.-A. Lázaro-Camí and J.-P. Ortega, Stochastic Hamiltonian dynamical systems, *Rep. Math. Phys.*, **61**(1), 65–122 (2008).

[53] J.-A. Lázaro-Camí and J.-P. Ortega, Reduction, reconstruction, and skew-product decomposition of symmetric stochastic differential equations, *Stoch. Dyn.*, textbf9(1), 1–46, (2009).

[54] J.-A. Lázaro-Camí and J.-P. Ortega, Superposition rules and stochastic Lie-Scheffers systems, *Ann. Inst. Henri Poincaré Probab. Stat.*. **45**(4), 910–931 (2009).

[55] J.-A. Lázaro-Camí and J.-P. Ortega, The stochastic Hamilton-Jacobi equation, *J. Geom. Mech.*, **1**(3), 295315 (2009).

[56] A. Lew, J. E. Marsden, M. Ortiz, and M. West, Variational time integrators, *Int. J. Numer. Methods Eng.*, **60**(1), 153–212 (2004).

[57] A. Lew, J. E. Marsden, M. Ortiz, and M. West, An overview of variational integrators, in *Finite Element Methods: 1970s and Beyond, CIMNE*, L. P. Franca, T. E. Tezduyar, A. Masud, eds., 98–115 (2004).

[58] S. Leyendecker, J. E. Marsden, and M. Ortiz, Variational integrators for constrained dynamical systems, *J. Appl. Math. Mech.*, **88**(9), 677–708 (2008).

[59] S. Leyendecker and S. Ober-Blöbaum, A variational approach to multirate integration for constrained systems, in *ECCOMAS Thematic Conference: Multibody Dynamics: Computational Methods and Applications, Brussels, Belgium, 47 July 2011*, P. Fisette, J.-C. Samin eds.

[60] S. Leyendecker, S. Oberblöbaum, J. E. Marsden, and M. Ortiz, Discrete mechanics and optimal control for constrained systems, *Optim. Control Appl. Methods*, **31**(6), 505–528 (2010)

[61] J. E. Marsden, D. Ebin, and A. E. Fischer, Diffeomorphism groups, hydrodynamics and relativity, *Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress, Differential Geometry and Applications, Dalhousie Univ., Halifax, N. S., 1971*, Vol. 1, 135–279, Canad. Math. Congr., Montréal, Quebec, 1972.

[62] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems*, second ed., Texts in Applied Mathematics, **17**, Springer-Verlag, New York, 1999.

[63] J. E. Marsden and J. Scheurle, The reduced Euler-Lagrange equations, *Fields Institute Comm.*, **1**, 139-164 (1993).

[64] J. E. Marsden and M. West, Discrete mechanics and variational integrators, *Acta Numer.*, **10**, 357–514 (2001).

[65] G. Misiołek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, *Indiana Univ. Math. Journ.*, **42**(1), 215–235 (1993).

[66] J. Moser and A. P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, *Commun. Math. Phys.*, **139**, 217–243 (1991).

[67] L. Noakes and T. S. Ratiu, Bi-Jacobi fields and Riemannian cubics for left invariant $SO(3)$, *Comm. Math. Sci.*, to appear.

[68] S. Ober-Blöbaum, O. Junge, and J. E. Marsden, Discrete mechanics and optimal control: an analysis, *Control Optim. Calc. Var.*, **17**(2), 322–352 (2011).

[69] H. Poincaré, Sur une forme nouvelle des équations de la mécanique, *C.R. Acad. Sci.*, **132**, 369–371 (1901).

[70] E. R. Priest, *Solar Magneto-Hydrodynamics*, Geophysics and Astrophysics Monographs, Volume 21, Springer-Verlag (1982).

[71] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.*, **36**(5), 635–664 (1983).

[72] Y. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*. Progress in Mathematics, **219**, Birkhäuser Verlag, Basel, 2003.

[73] M. Tao, H. Owhadi, and J. E. Marsden, Nonintrusive and structure preserving multiscale integration of stiff ODEs, SDEs, and Hamiltonian systems with hidden slow dynamics via flow averaging, *Multiscale Model. Simul.*, **8**(4), 1269–1324 (2010).

[74] K. Yasue, Stochastic calculus of variations, *J. Funct. Anal.*, **41**(3), 327–340 (1981).

[75] K. Yasue, A variational principle for the Navier-Stokes equation, *J. Funct. Anal.*, **51**(2), 133–141 (1983).

[76] J.-C. Zambrini, The research program of Stochastic Deformation (with a view toward Geometric Mechanics), *Stochastic Analysis: a series of lectures*, Springer, Progress in Probability 68, ed. R.C. Dalang, M. Dozzi, F. Flandoli, F. Russo (2015).